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Journal of Geometry and Physics 58 (2008) 576–598

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Bi-branes: Target space geometry for world sheet topological defects

Jürgen Fuchs[a], Christoph Schweigert[b], Konrad Waldorf[b,*]

[a] *Teoretisk fysik, Karlstads Universitet, Universitetsgatan 5, S – 651 88 Karlstad, Sweden*
[b] *Organisationseinheit Mathematik, Schwerpunkt Algebra und Zahlentheorie, Universität Hamburg, Bundesstraße 55, D – 20146 Hamburg, Germany*

Received 3 May 2007; accepted 19 December 2007
Available online 4 January 2008

Abstract

We establish that the relevant geometric data for the target space description of world sheet topological defects are submanifolds – which we call bi-branes – in the product $M_1 \times M_2$ of the two target spaces involved. Very much like branes, they are equipped with a vector bundle, which in backgrounds with non-trivial B-field is actually a twisted vector bundle. We explain how to define Wess–Zumino terms in the presence of bi-branes and discuss the fusion of bi-branes.

In the case of WZW theories, symmetry preserving bi-branes are shown to be biconjugacy classes. The algebra of functions on a biconjugacy class is shown to be related, in the limit of large level, to the partition function for defect fields. We finally indicate how the Verlinde algebra arises in the fusion of WZW bi-branes.

PACS: 11.25 HF; 11.25 Uv

Keywords: Bi-brane; Biconjugacy class; Bimodule; Fusion; Topological defect

1. Introduction

Sigma models have been a significant source of examples for two-dimensional conformal field theories. They allow one to relate geometric structure on target space to field theoretic quantities in the conformal field theory. This has provided much insight, not least for the interpretation of string theory. A particularly important observation has been the relationship between (conformal) world sheet boundary conditions and D-branes, which are, in their simplest incarnation, submanifolds of the target space equipped with a vector bundle.

The target space of a sigma model has, at least, the structure of a (pseudo-)Riemannian manifold. Further structure on the target space is introduced by the presence of the tachyon and of the antisymmetric Kalb–Ramond B-field. While we will ignore the tachyon in the present article, we do take the B-field into account, wherever this is possible without rendering the exposition too technical. The appropriate geometric structure on target space needed to describe a non-trivial B-field background is a hermitian bundle gerbe, and for a D-brane the vector bundle gets replaced by a twisted vector bundle, i.e. by a gerbe module for the restriction of the gerbe to the world volume of the brane.

* Corresponding author.
E-mail address: konrad.waldorf@math.uni-hamburg.de (K. Waldorf).

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.12.009

Apart from conformally invariant boundary conditions, two-dimensional conformal field theories admit another, equally natural, structure: topological defect lines. These objects are familiar from statistical mechanics. Take, for example, the lattice version of the Ising model: changing the coupling along all bonds that cross a specified line from ferromagnetic to antiferromagnetic produces a defect. Due to the $\mathbb{Z}_2$-gauge invariance of the Ising model, the position of this defect can be moved around, as long as we do not cross the site of a spin that appears in the correlator of interest. If we do cross such a site, we are forced to change the sign of the spin variable. The defect thus comes with a well-defined rule for passing insertions in the bulk through the defect line.

Moreover, in the Ising model a pair of two such defect lines which run close to each other can be eliminated by a gauge transformation; more generally, two defects can be joined to a single defect, which gives rise to fusion rules between topological defects. A similar phenomenon arises when we take boundary conditions into account: In the Ising model, a given boundary condition, say "spin up", combined with a parallel antiferromagnetic defect line can be replaced by the boundary condition "spin down". More generally, there is a mixed fusion by which topological defects act on conformal boundary conditions.

Like in the case of boundary conditions, in the CFT that is obtained in the continuum limit this structure can be expected to result in defect lines along which correlation functions of bulk fields can have a branch-cut like behaviour. At least for rational conformal field theories, such defect lines appear naturally in algebraic approaches to CFT [26,17]; in the TFT approach to RCFT correlators [17] a complete description of such defects is available [13,14]. The TFT approach allows one, in particular, to compute the partition functions of bulk and boundary fields, and of defect fields (fields living on a defect line that can change the type of defect), as well as the fusion of two defects and of a defect with a conformal boundary condition.

More specifically, suppose a collection of conformal field theories is *compatible* in the sense that they share a chiral symmetry algebra, including at least the Virasoro algebra. Note that in order for two conformal field theories to be compatible, they must in particular have the same Virasoro anomaly. Standard examples of compatible theories are the WZW models based on $SU(2)$ and on $SO(3)$ with the same value of the level. We label the members of a compatible family of conformal field theories by indices $\{A_1, A_2, \ldots\}$. There then exist (oriented) defects which separate the conformal field theory of type A_1 present on a region of world sheet to their left from a conformal field theory of type A_2 to their right hand side. Such a topological defect will be denoted by ${}_{A_1}B_{A_2}$. Then the fusion of defects associates with two defects ${}_{A_1}B_{A_2}$ and ${}_{A_2}B_{A_3}$ a defect of type ${}_{A_1}B_{A_3}$:

$$ {}_{A_1}B_{A_3} = {}_{A_1}B_{A_2} \star_{A_2} {}_{A_2}B_{A_3} \,. \tag{1} $$

The second type of fusion associates with a defect ${}_{A_1}B_{A_2}$ and boundary condition ${}_{A_2}N$ for the theory of type A_2 a boundary condition ${}_{A_1}N$ for the theory of type A_1,

$$ {}_{A_1}N = {}_{A_1}B_{A_2} \star_{A_2} {}_{A_2}N \,. \tag{2} $$

In the framework of [17,14], the labels $\{A_1, A_2, \ldots\}$ correspond to certain algebras in the representation category of the chiral symmetry algebra. These algebras encode in particular the partition functions, including a modular invariant bulk partition function and partition functions for boundary and defect fields. Branes are described by modules, and defects by bimodules, of these algebras; the fusion operation $\star_A$ is realized as the tensor product over A.

It has also been understood [13,14] that topological defects encode information both on internal symmetries and on dualities of a conformal field theory; this includes in particular T-dualities.

In view of the relevance of target space structures for string theoretic interpretations, it is natural to ask whether a target space description exists for conformal defects as well. The answer to this question is the primary result of the present paper.

Suppose we are given two compatible conformal field theories, corresponding to target spaces M_1 and M_2. We show that conformal defects correspond to submanifolds of the product $M_1 \times M_2$. Furthermore, very much in the same way as for a brane, this submanifold has to be endowed with a vector bundle (again, in the presence of a non-trivial B-field this is a twisted vector bundle). For theories based on current algebras – compactified free bosons and Wess–Zumino–Witten theories – we study the relevant submanifolds in detail. For simplicity, in this paper we restrict our attention to the cases of a single compactified free boson and of the WZW model based on a compact, connected and simply connected Lie group. It is clear, however, that when combined with standard techniques developed for

D-branes, the concepts presented here allow us to extend our results to more general classes of conformal field theories, in particular to WZW theories on non-simply connected groups, coset theories, theories of several free bosons compactified on a torus, and orbifolds of such theories.

In the rest of this paper we will proceed as follows. Inspired by the calculation of the scattering of closed string states in the presence of D-branes [7,8], in Section 2 we analyze scattering processes in the presence of defect lines, considering theories with current symmetries and defects of type ${}_A B_A$. In these cases we have $M_1 = M_2 = M$, and the target space M is a compact connected Lie group. In the simply connected case the relevant submanifold of $M \times M$ turns out to be a biconjugacy class, i.e. is of the form

$$\mathcal{B}_{h_1,h_2} := \left\{ (g_1, g_2) \in G \times G \mid \exists\, x, y \in G : g_1 = x h_1 y^{-1}, g_2 = x h_2 y^{-1} \right\}. \tag{3}$$

This is analogous to the role played by conjugacy classes [2,8,27] in the description of boundary conditions. Correspondingly, the so-called 2-characters

$$\chi^{(2)}_\lambda : \quad \begin{array}{l} G \times G \to \mathbb{C} \\ (g_1, g_2) \mapsto \mathrm{tr}_{H_\lambda}(g_1 g_2^{-1}) \end{array} \tag{4}$$

take the role that characters play in the theory of branes. We will therefore refer to the target space objects that describe defects as *bi-branes*.

It should be appreciated that while the multiplication of the Lie group G enters in the specific form of bi-branes for WZW theories, the description of defects in general does *not* require a multiplication on target space. Rather, the relevant structure for bi-branes separating theories with target spaces M_1 and M_2 are suitable submanifolds of $M_1 \times M_2$.

In Section 3 we discuss the intrinsic geometry of biconjugacy classes and relate the algebra of functions on a biconjugacy class to the algebra of defect fields; we can then exhibit a 2-form on the biconjugacy class that trivializes the difference of the 3-form field strengths on the two backgrounds involved. In Section 4 we show how these data can be employed to construct a Wess–Zumino term in situations in which the topologies of the target space and the bi-brane are particularly simple; a proof that the so constructed Wess–Zumino term is well-defined, as well as the description of the Wess–Zumino term for more general target spaces and/or bi-branes, is relegated to appendices. Finally, Section 5 is devoted to aspects of the fusion of two bi-branes and of the fusion of a bi-brane to a brane; we provide in particular an argument for how the Verlinde algebra arises as the fusion algebra of symmetry preserving bi-branes on simply connected Lie groups. A short outlook is supplied in Section 6.

2. Scattering of bulk fields in the backgrounds of defects

One rationale for assigning a target space geometry to a conformal field theory is to study the scattering of bulk fields. This is based on the general idea (see e.g. [15]) that (a subspace of) the space of bulk fields can be identified with a truncation and deformation of the algebra of functions on the target space. In the case of branes this amounts, in tree level approximation to string theory scattering amplitudes, to computing the two-point functions of bulk fields on a disk with given boundary condition. By factorization to a three-point function on the sphere and a one-point function on the disk, this can be reduced [7,8] to the computation of one-point functions of bulk fields on the disk.

Here we are interested in probing the target space geometry for a topological defect B on the world sheet, again using the scattering of bulk fields. In tree level approximation we have to consider the two-point functions of bulk fields on a world sheet that is a sphere S^2 containing a closed defect line B. Without loss of generality, we can take the defect line to be along the equator of the sphere. If both bulk field insertions are on the same hemisphere, then by factorization we just obtain the correlator in the absence of a defect, multiplied by the quantum dimension of the defect [10]. To get information on the relevant geometry of the target space, we must thus consider the situation with the two bulk insertions on different hemispheres, i.e. on different sides of the defect line.

For theories with current symmetry we will use the following notation. By $\bar{\mathfrak{g}}$ we denote a finite-dimensional reductive complex Lie algebra. Special cases of particular interest are those where $\bar{\mathfrak{g}}$ is simple, and the abelian Lie algebra $\mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1)$. By G we denote the simplest compact Lie group with Lie algebra (the compact real form of) $\bar{\mathfrak{g}}$. Thus for semisimple $\bar{\mathfrak{g}}$, G is the connected simply connected compact Lie group with Lie algebra $\bar{\mathfrak{g}}$, while for reductive Lie algebras we take in addition the direct product with d copies of $U(1)$, with d the dimension of the center of $\bar{\mathfrak{g}}$. For concreteness, the reader might wish to keep in mind the two special cases $\bar{\mathfrak{g}} = \mathfrak{u}(1)$ and $\bar{\mathfrak{g}} = \mathfrak{su}(2)$, with $G = U(1)$ and $G = SU(2)$, respectively.

By $\mathfrak{g}$ we denote the non-trivial central extension of the loop algebra of $\bar{\mathfrak{g}}$; if $\bar{\mathfrak{g}}$ is simple, $\mathfrak{g}$ is an untwisted affine Lie algebra, while for $\bar{\mathfrak{g}}$ abelian we have a direct sum of Heisenberg algebras with identified centers. We fix the value of the level k for each simple ideal of $\bar{\mathfrak{g}}$; the irreducible highest weight representations are then classified by the set P_k of dominant integral weights λ at level k. Analogously the irreducible finite-dimensional representations of $\bar{\mathfrak{g}}$ are labeled by the set P of dominant integral $\bar{\mathfrak{g}}$-weights. In particular, for $\bar{\mathfrak{g}} = \mathfrak{u}(1)$, Fock spaces are labeled by momentum, so that $P_k = P = \mathbb{R}$, while for $\bar{\mathfrak{g}} = \mathfrak{su}(2)$, at positive integral level k the relevant sets are $P_k = \{0, 1, \ldots, k\}$ and $P = \mathbb{Z}_{\geq 0}$.

Thus for any $\lambda \in P$ we have a finite-dimensional $\bar{\mathfrak{g}}$-module H_λ (for $\bar{\mathfrak{g}} = \mathfrak{su}(2)$ its dimension is $\lambda + 1$). We may as well regard H_λ as a G-module; its character is

$$\begin{aligned} \chi_\lambda: \quad & G \to \mathbb{C}^\times \\ & g \mapsto \mathrm{tr}_{H_\lambda} R_\lambda(g). \end{aligned} \tag{5}$$

Via taking the horizontal part of an affine weight, we can regard P_k as a subset of P. The irreducible $\mathfrak{g}$-module with highest weight $\lambda \in P_k$ is infinite-dimensional, with finite-dimensional homogeneous subspaces; we identify its zero-grade subspace with the finite-dimensional $\bar{\mathfrak{g}}$-module H_λ. Finally, by λ^+ we denote the highest weight of the representation that is conjugate to H_λ. For $\bar{\mathfrak{g}} = \mathfrak{u}(1)$, this is the representation with opposite $\mathfrak{u}(1)$-charge; for $\bar{\mathfrak{g}} = \mathfrak{su}(2)$, every representation is self-conjugate.

Returning to our preceding discussion, we now consider the correlation function on S^2 of two bulk fields labeled by $\mathfrak{g} \oplus \mathfrak{g}$-modules $H_\lambda \boxtimes H_{\lambda^+}$ and $H_\mu \boxtimes H_{\mu^+}$ inserted, respectively, at the north and south poles of S^2, with a defect B along the equator. Further, we restrict our attention to the so-called Cardy case, in which the bulk partition function is given by charge conjugation, boundary conditions are labeled by primary fields and the annulus coefficients are fusion rules [6]. In the Cardy case also the topological defects are labeled by the same set P_k as the left- and right-moving parts of the bulk fields. In the sequel we abbreviate the defect $B = B_\alpha$ with $\alpha \in P_k$ as α.

By holomorphic factorization, any correlator on S^2 is an element of the space of conformal blocks on the double cover of S^2, which consists of the disjoint union of two copies of $\mathbb{CP}^1$ with opposite orientation. For the correlator $\mathcal{D}_{\alpha;\lambda\mu}$ of two bulk fields on S^2 with a defect line α, we thus deal with a four-point block $D_{\lambda\mu}$ on $\mathbb{CP}^1 \sqcup \mathbb{CP}^1$, which is an element of the algebraic dual of the tensor product vector space $H_\lambda \otimes H_{\lambda^+} \otimes H_\mu \otimes H_{\mu^+}$. Like in [8] we consider the particular correlator

$$\mathcal{G}^{abcd}_{\alpha;\lambda\mu}(v \otimes \tilde{v} \otimes w \otimes \tilde{w}) := \mathcal{D}_{\alpha;\lambda\mu}(J^a_{-1} v \otimes J^b_{-1} \tilde{v} \otimes J^c_{-1} w \otimes J^d_{-1} \tilde{w}), \tag{6}$$

where by J^a_n, with a a labeling a basis of $\bar{\mathfrak{g}}$, we denote the modes of the currents $J^a(z)$ (for the corresponding basis elements of $\bar{\mathfrak{g}}$ we write $\bar{J}^a$).

In order for the correlator (6) to be non-zero we need $\mu = \lambda^+$. The states v and $\tilde{w}$ are then vectors in the $\bar{\mathfrak{g}}$-module H_λ, while $\tilde{v}$ and w are states in the $\bar{\mathfrak{g}}$-module H_{λ^+}, with these $\bar{\mathfrak{g}}$-modules regarded as the zero-grade subspaces of the corresponding $\mathfrak{g}$-modules.

To determine the correlation function (6), we first study the four-point conformal blocks $D_{\lambda\lambda^+}$ on $\mathbb{CP}^1 \sqcup \mathbb{CP}^1$. They decompose into a tensor product of two-point blocks on the two copies of $\mathbb{CP}^1$, $D_{\lambda\lambda^+} = F_\lambda \otimes F_{\lambda^+}$. The chiral Ward identities for left and right movers read

$$D_{\lambda\lambda^+} \circ \left(J^a_{-n} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes J^a_n \otimes \mathbf{1} \right) = 0 \tag{7}$$

and

$$D_{\lambda\lambda^+} \circ \left(\mathbf{1} \otimes J^a_{-n} \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes J^a_n \right) = 0, \tag{8}$$

respectively, for all $a = 1, 2, \ldots, \dim(\bar{\mathfrak{g}})$ and all $n \in \mathbb{Z}$. Together with the highest weight properties of w and $\tilde{w}$ and with the commutation relations of $\mathfrak{g}$, the Ward identities imply

$$\begin{aligned} & D_{\lambda\lambda^+}(J^a_{-1} v \otimes J^b_{-1} \tilde{v} \otimes J^c_{-1} w \otimes J^d_{-1} \tilde{w}) = D_{\lambda\lambda^+}(v \otimes \tilde{v} \otimes J^a_1 J^c_{-1} w \otimes J^b_1 J^d_{-1} \tilde{w}) \\ & \quad = F_\lambda(v \otimes [J^a_1, J^c_{-1}] w) F_{\lambda^+}(\tilde{v} \otimes [J^b_1, J^d_{-1}] \tilde{w}) \\ & \quad = \left[F_\lambda(v \otimes [\bar{J}^a, \bar{J}^c] w) + k\kappa^{ac} F_\lambda(v \otimes w) \right] \cdot \left[F_{\lambda^+}(\tilde{v} \otimes [\bar{J}^b, \bar{J}^d] \tilde{w}) + k\kappa^{bd} F_{\lambda^+}(\tilde{v} \otimes \tilde{w}) \right]. \end{aligned} \tag{9}$$

We expect that a direct contact to the geometry of compact Lie groups exists in the weak coupling limit, i.e. in the limit of large level k. Accordingly we only keep those terms in (9) which are of leading order in k; they are proportional to the Killing form of $\bar{\mathfrak{g}}$ and correspond to graviton and dilaton scattering; if $\bar{\mathfrak{g}}$ is abelian, they are the only terms present. In this limit we obtain the expression

$$k^2\kappa^{ac}\kappa^{bd}F_\lambda(v\otimes w)F_{\lambda^+}(\tilde{v}\otimes\tilde{w}) =: k^2\kappa^{ac}\kappa^{bd}D^{\infty}_{\lambda\lambda^+}(v\otimes\tilde{v}\otimes w\otimes\tilde{w}). \tag{10}$$

As in [8], at this point we invoke the Peter–Weyl theorem, so as to identify the space $\bigoplus_{\lambda\in P_k} H_\lambda \boxtimes H_{\lambda^+}$ with a subspace of the space $\mathcal{F}(G)$ of functions on the Lie group G. This way, equation (10) allows us to associate with a defect a linear function on $\mathcal{F}(G)$, i.e. a distribution. Before computing this distribution, which essentially amounts to a Fourier transformation, we notice that while boundary conditions give a distribution on G, defects give a distribution on the product manifold $G\times G$. As a consequence, defects will be associated with submanifolds of $G\times G$. This also fits nicely with the philosophy behind the so-called folding trick [30], by which a conformal defect separating two conformal field theories $\mathfrak{C}_{A_1}$ and $\mathfrak{C}_{A_2}$ with the same conformal anomaly is related to a conformally invariant boundary condition in the product theory $\mathfrak{C}_{A_1}\times\overline{\mathfrak{C}_{A_2}}$.[1] It should be kept in mind, however, that in this article we are only concerned with topological defects, which constitute a specific subclass of conformal defects.

Let us now Fourier transform the result (10) according to the rules of [8], to obtain a distribution on $G\times G$. We first note that the Fourier transformation of a linear form D on the space $\bigoplus_{\lambda,\mu\in P} H_\lambda\boxtimes H_{\lambda^+}\boxtimes H_\mu\boxtimes H_{\mu^+}$ reads

$$\begin{aligned} D(v\otimes\tilde{v}\otimes w\otimes\tilde{w}) &= \int_{G\times G}\mathrm{d}g\,\mathrm{d}g'\,\widetilde{D}(g,g')^*\sum_{\lambda,\mu\in P}\langle\tilde{v}\otimes\tilde{w}|\,R_\lambda(g)\otimes R_\mu(g')\,|v\otimes w\rangle \\ &= \int_G\mathrm{d}g\sum_{\lambda\in P}\langle\tilde{v}|\,R_\lambda(g)\,|v\rangle\int_G\mathrm{d}g'\sum_{\mu\in P}\langle\tilde{w}|\,R_\mu(g')\,|w\rangle\,\widetilde{D}(g,g')^*, \end{aligned}\tag{11}$$

and that its inverse is given by

$$\widetilde{D}(g,g') = \sum_{\mu_1;i,j}\sum_{\mu_2;k,l} N_{\mu_1}N_{\mu_2}D(v_i\otimes\tilde{v}_j\otimes w_k\otimes\tilde{w}_l)\cdot\langle\tilde{v}_j|R_{\mu_1}(g)\,|v_i\rangle\,\langle\tilde{w}_l|\,R_{\mu_2}(g')\,|w_k\rangle\,, \tag{12}$$

with $\{v_i\}$ a basis of H_{μ_1} and $\{\tilde{v}_i\}$ the dual basis of $H_{\mu_1^+}$, and analogously for w_k and $\tilde{w}_k$. Here the normalization factors N_{μ_i} are given by $N_\mu=\sqrt{|H_\mu|/|G|}$ with $|H_\mu|$ the dimension of H_μ and $|G|$ the volume of G.[2]

For the functions (10) of interest to us this prescription yields

$$\begin{aligned} \widetilde{D}^{\infty}_{\lambda\lambda^+}(g,g') &= \sum_{\mu_1,\mu_2\in P}N_{\mu_1}N_{\mu_2}\sum_{i,j,k,l}\left\langle\tilde{v}_j\right|R_{\mu_1}(g)\,|v_i\rangle\cdot F_\lambda(v_i\otimes v_k)\,\langle\tilde{v}_l|\,R_{\mu_2}(g')\,|v_k\rangle\,F_{\lambda^+}(\tilde{v}_j\otimes\tilde{v}_l) \\ &= N_\lambda^2\sum_{i,j,k,l}\left\langle\tilde{v}_j\right|R_\lambda(g)\,|v_i\rangle\,F_\lambda(v_i\otimes v_k)\,\langle\tilde{v}_l|\,R_{\lambda^+}(g')\,|v_k\rangle\,F_{\lambda^+}(\tilde{v}_j\otimes\tilde{v}_l). \end{aligned}\tag{13}$$

By the identities $R_{\lambda^+}(g)=(R_\lambda(g^{-1}))^{\mathrm{t}}$, where the superscript indicates the transpose matrix, and $F_\lambda(v_i\otimes v_k)=\delta_{i,k}$, this reduces to

$$\widetilde{D}^{\infty}_{\lambda\lambda^+}(g,g') = N_\lambda^2\sum_{i,j}(R_\lambda(g))_i^{\,j}\left(R_\lambda(g'^{-1})\right)_j^{\,i} = N_\lambda^2\chi_\lambda(gg'^{-1}). \tag{14}$$

Here 2-characters of G pop up. 2-characters are functions on the Cartesian product $G\times G$ of a group with itself. They first appeared in [12] in the expansion of group determinants. As compared to characters, they contain more information about the group than characters; e.g. in contrast to characters, they allow one to determine whether a representation is real or pseudo-real. (Still, 2-characters and characters do not determine a group up to isomorphism. A surprisingly recent result [22] states that a finite group is determined by its 1-, 2- and 3-characters.)

[1] Indeed, from the way a biconjugacy class can be related to a permutation-twisted conjugacy class in $G\times G$ (as e.g. considered in [11]) one can deduce information about a possible target space implementation of the folding trick. We thank Thomas Quella for a discussion on this point.

[2] Note that, like e.g. in [3,19], we do not take the volume of G to be normalized to 1. Rather, the 'physical' radius of G should be $\sqrt{k\alpha'}$, i.e. $|G|$ is proportional to $(k\alpha')^{\dim(G)/2}$.

Next we use the results of the TFT approach (following the lines of Section 4 of [18]) to express the correlation functions in terms of conformal blocks: we have

$$\mathcal{D}_{\alpha;\lambda\lambda^+} = \frac{S_{\lambda,\alpha}}{S_{0,\lambda}} D_{\lambda\lambda^+} = \chi_\alpha(h_\lambda)^* D_{\lambda\lambda^+} = \frac{S_{0,\alpha}}{S_{0,\lambda}} \chi_\lambda(h_\alpha)^* D_{\lambda\lambda^+}, \tag{15}$$

where like in [8] we introduced the group element

$$h_\alpha := \exp(2\pi\mathrm{i}\hat{y}_\alpha), \tag{16}$$

with $\hat{y}_\alpha$ the Cartan subalgebra element dual to the weight

$$y_\alpha := \frac{\alpha+\rho}{k+g^\vee} \in \overline{\mathfrak{g}}_0^*. \tag{17}$$

(ρ denotes the Weyl vector and $g^\vee$ the dual Coxeter number of $\overline{\mathfrak{g}}$.) For the sum

$$\mathcal{G}_\alpha^{abcd} := \sum_{\lambda\in P_k} \mathcal{G}_{\alpha;\lambda\lambda^+}^{abcd} \tag{18}$$

of two-point correlators, which is the analogue of a boundary state, we thus obtain, at large k,

$$\widetilde{G}_\alpha^{abcd}(g,g') = k^2\kappa^{ac}\kappa^{bd} \sum_{\lambda\in P_k} N_\lambda^2 \frac{S_{0,\alpha}}{S_{0,\lambda}} \chi_\lambda(h_\alpha)^* \chi_\lambda(g{g'}^{-1}). \tag{19}$$

Furthermore, using that at large k the quantum dimension $S_{0,\lambda}/S_{0,0}$ approaches the ordinary dimension $|H_\lambda|$ and P_k can be replaced by P, this reduces to

$$\widetilde{G}_\alpha^{abcd}(g,g') = k^2\kappa^{ac}\kappa^{bd} \frac{|H_\alpha|}{|G|} \sum_{\lambda\in P} \chi_\lambda(h_\alpha)^* \chi_\lambda(g{g'}^{-1}). \tag{20}$$

Up to normalization this is a delta distribution on the conjugacy class $\mathcal{C}_\alpha \equiv \mathcal{C}_{h_\alpha}$ of G:

$$\sum_{\lambda\in P} \chi_\lambda(h_\alpha)^* \chi_\lambda(g{g'}^{-1}) = \frac{|G|}{|\mathcal{C}_\alpha|} \delta_{\mathcal{C}_\alpha}(g{g'}^{-1}). \tag{21}$$

Thus we finally arrive at

$$\widetilde{G}_\alpha^{abcd}(g,g') \xrightarrow{k\to\infty} k^2\kappa^{ac}\kappa^{bd} \frac{|H_\alpha|}{|\mathcal{C}_\alpha|} \delta_{\mathcal{C}_\alpha}(g{g'}^{-1}). \tag{22}$$

In short, for given topological defect α, in the large level limit the analogue (18) of the boundary state is concentrated on those pairs $(g,g') \in G\times G$ whose product $g{g'}^{-1}$ lies in $\mathcal{C}_\alpha$.

3. The world volume of WZW bi-branes

3.1. Biconjugacy classes

According to the scattering calculation in the previous section, the geometric object in $G\times G$ that is relevant for the description of a defect α is the set of those points (g_1,g_2) of $G\times G$ such that $g_1g_2^{-1}$ lies in the conjugacy class $\mathcal{C}_\alpha$ of G. These subsets of $G\times G$ are actually submanifolds; we wish to describe them in more detail. To this end we introduce the following notion: For a compact connected Lie group G and elements $h_1,h_2\in G$ we call the submanifold

$$\mathcal{B}_{h_1,h_2} := \left\{(g_1,g_2)\in G\times G \mid \exists x_1,x_2\in G : g_1 = x_1h_1x_2^{-1},\, g_2 = x_1h_2x_2^{-1}\right\} \tag{23}$$

of $G\times G$ the *biconjugacy class* of the pair (h_1,h_2).

Biconjugacy classes inherit from the diagonal left and diagonal right actions of G on $G\times G$ two commuting actions of G. For the defects that we are describing here, these two G-actions correspond to the two independent preserved current symmetries.

Obviously, 2-characters are constant on biconjugacy classes. In fact, very much like the characters of irreducible G-representations form a natural basis for the functions on the space of conjugacy classes, the 2-characters of irreducible representations form a basis for the space of functions on biconjugacy classes.

Next we observe that the smooth map

$$\begin{aligned} \tilde{\mu}: \quad & G\times G \to G \\ & (g_1, g_2) \mapsto g_1 g_2^{-1} \end{aligned} \tag{24}$$

intertwines the diagonal left and diagonal right action of G on $G\times G$ and the adjoint and trivial actions of G on itself, respectively. Put differently, $\tilde{\mu}$ defines the structure of a trivializable G-equivariant principal G-bundle over G. Indeed, the G-action on the fibers is by diagonal right multiplication, so that the G-equivariant diffeomorphism $t: (g_1, g_2) \mapsto (g_1 g_2, g_2)$ furnishes a global trivialization, where the trivial G-bundle $p_1: G\times G \to G$ over G projects on the first component.

It now follows that a biconjugacy class in $G\times G$ is the preimage of a conjugacy class in G under the projection $\tilde{\mu}$ defined in (24):

$$\mathcal{B}_{h_1,h_2} = \tilde{\mu}^{-1}(\mathcal{C}_{h_1 h_2^{-1}}) = \left\{(g_1, g_2) \in G\times G \mid g_1 g_2^{-1} \in \mathcal{C}_{h_1 h_2^{-1}}\right\}; \tag{25}$$

in particular,

$$\mathcal{B}_{h_1,h_2} = \mathcal{B}_{h_1 h_2^{-1},e}. \tag{26}$$

To establish the relation (25), we observe that for every element $(g_1, g_2) \in \mathcal{B}_{h_1,h_2}$ we have $g_1 = x_1 h_1 x_2^{-1}$ and $g_2 = x_1 h_2 x_2^{-1}$ for some $x_1.x_2 \in G$, and hence $g_1 g_2^{-1} = x_1 h_1 h_2^{-1} x_1^{-1} \in \mathcal{C}_{h_1 h_2^{-1}}$. Conversely, given $(g_1, g_2) \in G\times G$ such that there exists some $x \in G$ with $x g_1 g_2^{-1} x^{-1} = h_1 h_2^{-1}$, we set $x_1 := x^{-1}$ and $x_2 := g_2^{-1} x^{-1} h_2$ and obtain $g_1 = x_1 h_1 x_2^{-1}$ and $g_2 = x_1 h_2 x_2^{-1}$, which shows that $(g_1, g_2) \in \mathcal{B}_{h_1,h_2}$.

To conclude, biconjugacy classes have the topology of a direct product of G with a conjugacy class. Thus for simply connected groups, they are in particular simply connected. The scattering of closed string states in WZW theories detects bi-branes corresponding to biconjugacy classes for which $h_1 h_2^{-1}$ is a regular element of G; this closely parallels the findings of [8] for branes.

3.2. World volume quantization

As further evidence for the relation between biconjugacy classes and WZW defects, we will now establish that the defect fields associated with a topological defect furnish a quantization of the space of functions on a biconjugacy class. Note that besides bulk fields there also exist other types of fields in the presence of defects [14]: disorder fields, at which defect lines start or end, and defect fields, which live on a defect line and can change the type of the defect. There is a distinguished type of defect, acting as a unit with respect to fusion, called the invisible defect. Across this defect, every bulk field is smooth. Disorder fields are in fact special defect fields: those changing the invisible defect to some other defect or vice versa. Similarly, bulk fields can be regarded as defect fields preserving the invisible defect and thus as special disorder fields.

Since there are two commuting actions of G on the world volume of a biconjugacy class, the space $\mathcal{F}(\mathcal{B}_{h_1,h_2})$ has the structure of a $G\times G$-module. This can be compared with the situation for conjugacy classes, which describe WZW branes. A conjugacy class $\mathcal{C}$ carries a natural G-action, the adjoint action, which turns the space of $\mathcal{F}(\mathcal{C})$ of functions on $\mathcal{C}$ into a G-module. As pointed out in [8], only regular conjugacy classes are relevant to the situation of interest to us. A regular conjugacy class is isomorphic to G/T, with T a maximal torus of G, and there is an isomorphism

$$\mathcal{F}(G/T) \cong \bigoplus_{\lambda\in P} \mathrm{mult}_\lambda(0)\, H_\lambda \tag{27}$$

of G-modules, where $\mathrm{mult}_\lambda(0)$ denotes the multiplicity of the weight 0 in the highest weight $\bar{\mathfrak{g}}$-module H_λ.

This G-module structure is related, in the large level limit, to the G-module structure of a subset of the space of boundary fields for the corresponding WZW brane. Note that in the present context we should take the large-k limit in a way such that the geometric conjugacy class is kept fixed. As a consequence, the weight labeling the boundary condition depends on the level. More specifically, just like in [8] we must consider weights $\alpha = \alpha(k)$ such that

$$\alpha_0 := \frac{\alpha(k)+\rho}{k+g^\vee} \tag{28}$$

is constant. The large-k limit of the WZW annulus coefficients $\mathrm{A}_{\lambda\alpha}^{\beta}$ for the case of simply connected G reads [8]

$$\lim_{k\to\infty} {}^{(k)}\mathrm{A}_{\lambda\alpha(k)}^{\beta(k)} = \delta_{\alpha_0,\beta_0}\,\mathrm{mult}_\lambda(0). \tag{29}$$

This result can be interpreted as follows. In the large level limit, only open strings starting and ending at the same brane survive. As a G-module, they have the algebra of functions on the brane as a limit; this substantiates the idea that the space of open strings constitutes a quantization of the world volume of the brane.

For bi-branes, we can obtain an analogous result by using $G \times G$-modules in place of G-modules. To describe the intrinsic geometry of the bi-brane $\mathcal{B}_{h_1,h_2}$, with h_1 and h_2 regular elements of G, we first note that the bijection

$$\begin{aligned}(p_1 \times \tilde{\mu}):\quad & \mathcal{B}_{h_1,h_2} \to G \times \mathcal{C}_{h_1h_2^{-1}} \\ & (g_1,g_2) \mapsto (g_1, g_1g_2^{-1})\end{aligned} \tag{30}$$

intertwines two pairs of G-actions: first, the diagonal left action of G on $\mathcal{B}_{h_1,h_2}$, i.e. $\rho(h)((g_1,g_2)) = (hg_1, hg_2)$, is intertwined with G acting from the left on itself and by the adjoint action on $\mathcal{C}_{h_1h_2^{-1}}$; and second, the diagonal right action on $\mathcal{B}_{h_1,h_2}$ is intertwined with the right action on G and the trivial action on $\mathcal{C}_{h_1h_2^{-1}}$. The $G\times G$-module structure of the space of functions on $\mathcal{B}_{h_1,h_2}$ now follows easily; we have

$$\mathcal{F}(\mathcal{B}_{h_1,h_2}) \cong \mathcal{F}(G \times \mathcal{C}_{h_1h_2^{-1}}) \cong \mathcal{F}(G) \otimes \mathcal{F}(\mathcal{C}_{h_1h_2^{-1}}). \tag{31}$$

Further, by the Peter–Weyl theorem we have $\mathcal{F}(G) \cong \bigoplus_{\mu\in P} H_\mu \boxtimes H_{\mu^+}$, while the G-module structure of $\mathcal{F}(\mathcal{C}_{h_1h_2^{-1}})$ is given by (27). Thus after decomposing the tensor product we obtain

$$\mathcal{F}(\mathcal{B}) \cong \bigoplus_{\lambda,\mu\in P} \left(\sum_{\nu\in P} \overline{\mathcal{N}}_{\nu\mu^+}^{\ \lambda}\,\mathrm{mult}_\nu(0) \right) H_\lambda \boxtimes H_\mu, \tag{32}$$

where $\overline{\mathcal{N}}_{\nu\mu^+}^{\ \lambda}$ is the multiplicity of the irreducible $\bar{\mathfrak{g}}$-module H_λ in the tensor product $H_\nu \otimes H_{\mu^+}$.

The decomposition (32) has to be compared with the multiplicities $Z_{\mu\nu}^{\alpha\beta}$ for defect fields with chiral labels μ,ν that change a defect α to a defect β. A simple calculation in the TFT approach to rational conformal field theories (compare Section 5.10 of [17]) shows that, in the Cardy case, this multiplicity is an ordinary fusion rule. Accordingly, we have at level k

$${}^{(k)}Z_{\lambda\mu}^{\alpha(k)\beta(k)} = {}^{(k)}\mathcal{N}_{\lambda\mu\alpha(k)}^{\beta(k)} \equiv \sum_{\nu\in P_k} {}^{(k)}\mathcal{N}_{\lambda\mu}^{\nu}\ {}^{(k)}\mathcal{N}_{\nu\alpha(k)}^{\beta(k)}\,. \tag{33}$$

The large-k limit of the two factors in this result follows easily: the fusion rules ${}^{(k)}\mathcal{N}_{\lambda\mu}^{\nu}$ tend to tensor product multiplicities, while the limit of the second factor is the same as the one computed above for the annulus coefficients (which for the Cardy case coincide with ordinary fusion rules). Thus we find

$$\begin{aligned}\lim_{k\to\infty} {}^{(k)}Z_{\lambda\mu}^{\alpha(k)\beta(k)} &= \delta_{\alpha_0,\beta_0} \sum_{\nu\in P} \overline{\mathcal{N}}_{\lambda\mu}^{\ \nu}\,\mathrm{mult}_\nu(0) \\ &= \delta_{\alpha_0,\beta_0} \sum_{\nu\in P} \overline{\mathcal{N}}_{\nu\mu^+}^{\ \lambda}\,\mathrm{mult}_\nu(0),\end{aligned} \tag{34}$$

where in the second equality the charge conjugation properties of the tensor product multiplicities are used. This is in full agreement with the $G\times G$-module structure (32) of the space $\mathcal{F}(\mathcal{B})$ of functions on the bi-brane. Analogously, as for branes, this substantiates the idea that the algebra of defect fields can be regarded as a quantization of the space of functions on the bi-brane.

3.3. Trivialization of the H-field

As is well known [29], conformal invariance for theories with non-abelian currents requires a non-trivial B-field background. While the B-field is defined only locally, its curvature H is a globally defined 3-form. One important property of branes is the fact that the restriction of H to the corresponding submanifolds is exact. For symmetric branes in the WZW model based on $\mathfrak{g}$ at level k, the curvature is the 3-form

$$H = \frac{k}{6}\langle\theta \wedge [\theta \wedge \theta]\rangle, \tag{35}$$

where we have denoted by θ the left-invariant Maurer–Cartan form on G, which is a $\overline{\mathfrak{g}}$-valued 1-form, and by $\langle\cdot,\cdot\rangle$ the Killing form on $\overline{\mathfrak{g}}$. Restricted to a conjugacy class $\mathcal{C}_h$, the 3-form H can be written as the derivative of a G-invariant 2-form ω_h,

$$H|_{\mathcal{C}_h} = \mathrm{d}\omega_h. \tag{36}$$

We will now see that bi-branes have properties that generalize this behaviour.

Consider again the map $\tilde{\mu}$ whose restriction maps the bi-brane $\mathcal{B}_{h_1,h_2}$ to the conjugacy class $\mathcal{C}_{h_1h_2^{-1}}$. We introduce the 2-form

$$\varpi_{h_1,h_2} := \tilde{\mu}^*\omega_{h_1h_2^{-1}} - \frac{k}{2}\langle p_1^*\theta \wedge p_2^*\theta\rangle \tag{37}$$

on $\mathcal{B}_{h_1,h_2}$, where p_i, $i = 1, 2$, is the projection from $G \times G \to G$ on its ith factor, and both summands are restricted to the submanifold $\mathcal{B}_{h_1,h_2}$ of $G \times G$. From the intertwining properties of $\tilde{\mu}$ it follows that the 2-form ϖ is bi-invariant. Analogously to the equality (36) on the world volume of a brane, on the world volume $\mathcal{B}_{h_1,h_2}$ of the bi-brane the identity

$$p_1^*H = p_2^*H + \mathrm{d}\varpi_{h_1,h_2} \tag{38}$$

holds; in other words: on $\mathcal{B}_{h_1,h_2}$, the difference of the H-fields of the two target spaces involved is exact and equals the derivative of the 2-form ϖ_{h_1,h_2}.

To establish the identity (38), we first recall the relation

$$\tilde{\mu}^*H = p_1^*H - p_2^*H + \frac{k}{2}\mathrm{d}\langle p_1^*\theta \wedge p_2^*\theta\rangle \tag{39}$$

(compare e.g. the proof of Proposition 3.2 of [1]) which in the derivation of the Polyakov–Wiegmann formula accounts for the correct behaviour of the Wess–Zumino term. On the other hand, we find

$$\left(\tilde{\mu}^*H\right)|_{\mathcal{B}_{h_1,h_2}} = \tilde{\mu}^*(H|_{\mathcal{C}_{h_1h_2^{-1}}}) = \tilde{\mu}^*(\mathrm{d}\omega_{h_1h_2^{-1}}) = \mathrm{d}\tilde{\mu}^*\omega_{h_1h_2^{-1}}; \tag{40}$$

together with the definition of ϖ_{h_1,h_2} the last two equations imply (38).

At this point it is worth mentioning the notion of a quasi-Hamiltonian G-space which has been introduced in [1]. As shown in [1], both conjugacy classes and the "double" $G \times G$ are examples of such spaces. However, the reader should be warned that, while the case of conjugacy classes is directly relevant for the discussion of branes, the double as considered in [1] is endowed with a $G \times G$-action that does not restrict to the bi-brane submanifolds.

4. The Wess–Zumino term in the presence of defects

Having identified a 2-form ϖ on the bi-brane that trivializes the restriction of the difference of the H-fields, we are in a position to study the Wess–Zumino term for situations with particularly simple topology. The analysis closely parallels the one in [9]. As in the case of branes, a general and more satisfactory analysis must be based on the notion of hermitian bundle gerbes. A first discussion of these issues can be found in Appendix B.

To attain a situation with sufficiently simple topology, we restrict our attention in the sequel to 2-connected target spaces M_1 and M_2, i.e. besides being connected and simply connected, the manifolds M_i also satisfy $\pi_2(M_i) = 0$ (this includes in particular compact connected and simply connected simple Lie groups). Because a bundle gerbe over a 2-connected space is completely determined by its curvature, which is a closed 3-form with integral periods, we may then consider target spaces M_1 and M_2 with closed integral 3-forms H_1 and H_2.

A similar phenomenon occurs for bi-branes if we make the additional assumption that the world volume of a bi-brane is connected and simply connected: the 2-form ϖ that trivializes the difference of the 3-forms is a sufficient substitute for the structure that is needed in the general case as described in Appendix B. Note that all these assumptions are in particular met for WZW bi-branes of simply connected compact Lie groups.

Under these assumptions, we arrive at the following simplified definition of a bi-brane: A *simply connected* M_1–M_2*-bi-brane* between 2-connected target spaces M_1 and M_2 with 3-forms $H_i \in \Omega^3(M_i)$, $i = 1, 2$, is a simply connected submanifold Q of $M_1 \times M_2$ together with a 2-form $\varpi \in \Omega^2(Q)$ such that

$$p_1^* H|_Q = p_2^* H|_Q + \mathrm{d}\varpi. \tag{41}$$

The classical Wess–Zumino–Witten model is a theory of maps from a two-dimensional world sheet to a target space. The space of maps has to be chosen in a way conforming with the correlator of interest. For example, for world sheets with non-empty boundary it is required that the boundary of the world sheet is mapped into the world volume of a WZW brane. Here our aim is to describe correlators with defect lines. We merely consider the simplest situation: a closed oriented world sheet Σ with an embedded oriented circle $S \subset \Sigma$ that separates the world sheet into two components, $\Sigma = \Sigma_1 \cup_S \Sigma_2$, which we assume to inherit the orientation of Σ. Without loss of generality we assume $\partial\Sigma_1 = S$ and $\partial\Sigma_2 = \overline{S}$ as equalities of oriented manifolds, where $\overline{S}$ is the manifold S with opposite orientation.

We assume that the defect separates regions that support conformally invariant sigma models with target spaces M_1 and M_2 and consider pairs of maps

$$\phi_i : \Sigma_i \to M_i \tag{42}$$

such that the image of the combined map

$$\begin{aligned} \phi_S : \quad & S \to M_1 \times M_2 \\ & s \mapsto (\phi_1(s), \phi_2(s)) \end{aligned} \tag{43}$$

takes its values in the submanifold Q.

We next wish to find the Wess–Zumino part of the action. First, since Q is simply connected, there exists a two-dimensional oriented submanifold D of Q with $\partial D = \phi_S(S)$. We can glue the images of this disk under the projections $p_i : M_1 \times M_2 \to M_i$ along their boundaries on the images $\phi_i(\Sigma_i)$ of the world sheets, and obtain two-dimensional oriented closed submanifolds. Because we have required $\pi_2(M_i) = (0)$, we can fill those to three-dimensional oriented submanifolds $B_i \subset M_i$ such that

$$\partial B_1 = \phi_1(\Sigma_1) \cup p_1(\overline{D}) \quad \text{and} \quad \partial B_2 = \phi_2(\Sigma_2) \cup p_2(D). \tag{44}$$

Equipped with such choices of submanifolds, we define

$$S[\phi_1, \phi_2] := \int_{B_1} H_1 + \int_{B_2} H_2 + \int_D \varpi. \tag{45}$$

Note that superficially the expression (45) depends on the choices of the manifolds B_1, B_2 and D. However, the ambiguities are integers, so that the exponential of (45) is actually well-defined. This can be shown with the help of a homology theory based on two manifolds M_1 and M_2 and a submanifold $Q \subset M_1 \times M_2$, which we set up in Appendix A. For the dual cohomology theory a theorem of de Rham type holds; it allows us to express a cohomology class with values in $\mathbb{R}$ as a triple of differential forms. The triple (H_1, H_2, ϖ) then furnishes an example of a cocycle in this cohomology theory. As we show in Appendix A, the ambiguities of (45) arise as the pairing of the cohomology class of (H_1, H_2, ϖ) with a cycle in homology that results from different choices of the submanifolds D, B_1 and B_2. We then show that if the cocycle (H_1, H_2, ϖ) corresponds to a cohomology class with values in $\mathbb{Z}$ – we shall call such a triple integral – the ambiguities of (45) are integers.

This is analogous to the discussion of the Wess–Zumino term in the presence of branes [9]: in that case the relative cohomology of the pair (M, Q) is relevant, where Q is the world volume of the brane. The 3-form H and the 2-form ω on Q define a cocycle in the relative cohomology with values in $\mathbb{R}$, and the Wess–Zumino term is the pairing of (H, ω) with a certain cycle. Its well-definedness imposes the condition that (H, ω) is integral, i.e. lies in the cohomology with values in $\mathbb{Z}$. As in the case of branes, the integrality condition described above imposes severe restrictions on the biconjugacy classes that can describe defect lines. In fact, only those biconjugacy classes which are of the form $\tilde{\mu}^{-1}(\mathcal{C})$ qualify, where $\mathcal{C} \subset G$ is a suitable conjugacy class, namely one that supports a gerbe module which leads to a boundary condition preserving all chiral currents at level k. It should be appreciated, though, that the 2-form on the biconjugacy class differs from the pull-back of the 2-form on the conjugacy class, and in fact there is no sensible way in which a gerbe bimodule can be seen as the pull-back of a gerbe module.

In Appendix B we show how one can drop the restrictions $\pi_2(M_i) = \pi_1(M_i) = 0$ on the topology of the background and $\pi_1(Q) = 0$ on the topology of the bi-brane world volume. In the absence of these conditions, it is not enough any longer to work with the 2-form ϖ on the bi-brane and the curvature 3-forms H_i on the backgrounds. Rather, connection-type data must be taken into account. This can be achieved using hermitian bundle gerbes, together with a new notion to be introduced in Appendix B: *gerbe bimodules*. We refer to the same appendix for the definition of a Wess–Zumino term in this general situation. To show that the proposed Wess–Zumino term restores the conformal symmetry of correlators with defects is beyond the scope of this article.

5. Fusion of bi-branes

As pointed out in the introduction, there are two natural notions of fusion involving bi-branes: the fusion of two bi-branes, and the fusion of a bi-brane and a brane to a brane. In both cases, the fusion of elementary (bi-)branes yields, in general, a superposition of elementary (bi-)branes.

As has been seen in the algebraic approach, for WZW defects that preserve all current symmetries there exists a natural notion of duality. It can be characterized by the property that the fusion of a bi-brane and its dual contains the special bi-brane which with respect to fusion acts as the identity. Ignoring the shift in the location of bi-branes by the Weyl vector, this is the bi-brane whose world volume is the biconjugacy class $\mathcal{B}_{(e,e)}$, i.e. the diagonal $G \subset G \times G$. Upon quantization, the functions on this special bi-brane are related to ordinary bulk fields, rather than general defect fields.

By invoking this duality, instead of working with the fusion rules

$$\mathcal{B}_\alpha \star \mathcal{B}_\beta = \sum_\gamma \mathcal{N}_{\alpha\beta}^{\gamma} \mathcal{B}_\gamma \tag{46}$$

of bi-branes we sometimes consider the multiplicities

$$\mathcal{N}_{\alpha\beta\gamma} := \mathcal{N}_{\alpha\beta}^{\gamma^\vee}. \tag{47}$$

These structure constants are, in general, not symmetric; from the results of the algebraic approach, however, we expect them to be invariant under cyclic permutations. The algebraic approach also predicts that in the case of compact connected and simply connected Lie groups, the constants $\mathcal{N}_{\alpha\beta}^{\gamma}$ are just the ordinary fusion multiplicities arising in the chiral theory, which satisfy the Verlinde formula.

5.1. World volume fusion

We first consider the effect of fusion on world volumes. In this context, the notation becomes more transparent when considering at once bi-branes describing defects that separate different target spaces M_1 and M_2.

The action of correspondences on sheaves suggests considering the following prescription: For the fusion of an M_1–M_2-bi-brane with world volume $B \subseteq M_1 \times M_2$ and an M_2-brane with world volume $V \subseteq M_2$ one should consider

$$B \star V := p_1\left(B \cap p_2^{-1}(V)\right) \tag{48}$$

with p_i the ith projection $M_1 \times M_2 \to M_i$. In general $B \star V$ is only a subset, rather than a submanifold, of M_1. On a heuristic level one would expect, however, that the quantization of the branes [3] selects a finite superposition of branes, which then should reproduce the results obtained in the TFT approach. The quantization conditions on the positions of branes require additional geometric structure on the branes, namely twisted vector bundles, and involve a subtle interplay of this structure with the background B-field. We will exhibit in examples how the required finite superposition of branes or bi-branes arises after geometric quantization.

Similarly, the fusion of an M_1–M_2-bi-brane B with an M_2–M_3-bi-brane B' uses projections p_{ij} from the triple product $M_1 \times M_2 \times M_3$ to the twofold products $M_i \times M_j$:

$$B \star B' := p_{13}\left(p_{12}^{-1}(B) \cap p_{23}^{-1}(B')\right). \tag{49}$$

Again the question of quantization should be addressed. This issue turns out to be largely parallel to what happens in the mixed fusion of bi-branes to branes, and accordingly we will concentrate on the case of mixed fusion.

5.2. *Bi-branes of the compactified free boson at fixed radius*

We consider a free boson compactified on a circle S^1_R of radius R and restrict ourselves, for the moment, to defects separating two world sheet regions that support one and the same theory. In this situation, it does no harm to identify the circle with the Lie group $\mathrm{U}(1) \cong \{z \in \mathbb{C} \mid |z| = 1\}$.

We consider two types of branes: D0-branes $V^{(0)}_x$ are localized at the position $x \in \mathbb{R} \bmod 2\pi R\mathbb{Z}$. D1-branes, in contrast, wrap the whole circle. The D1-brane characterized by a Wilson line $\alpha \in \mathbb{R} \bmod \frac{1}{2\pi R}\mathbb{Z}$ will be denoted by $V^{(1)}_\alpha$; the Wilson line describes a flat connection on S^1_R.

The world volume of a bi-brane on S^1_R is a submanifold of $S^1_R \times S^1_R$ of the form

$$B_x := \{(y, y - x) \mid y \in \mathbb{R} \bmod 2\pi R\mathbb{Z}\} \tag{50}$$

with $x \in \mathbb{R} \bmod 2\pi R\mathbb{Z}$. B_x has the topology of a circle, and according to our general considerations in Appendix B it must be endowed with a flat connection, i.e. with a Wilson line α. As a consequence, the natural parameters for bi-branes of a compactified free boson are a pair (x, α) taking values in two dual circles describing a position on S^1 and a Wilson line. We will write $\mathcal{B}_{(x,\alpha)} \equiv (B_x, \alpha)$ for such bi-branes.

For the fusion of a bi-brane $\mathcal{B}_{(x,\alpha)}$ and a D0-brane $V^{(0)}_y$ we have

$$\begin{aligned} p_2^{-1}(V^{(0)}_y) \;&=\; \{(y', y) \mid y' \in [0, 2\pi R)\}, \qquad B_x \cap p_2^{-1}(V^{(0)}_y) = \{(x + y, y)\} \\ &\text{and} \quad p_1\left(B_x \cap p_2^{-1}(V^{(0)}_y)\right) = \{x + y\}, \end{aligned} \tag{51}$$

so that the prescription (48) yields

$$\mathcal{B}_{(x,\alpha)} \star V^{(0)}_y = V^{(0)}_{x+y}. \tag{52}$$

Thus the fusion with a defect of type $\mathcal{B}_{(x,\alpha)}$ acts on D0-branes as a translation by x in position space.

For the fusion of a bi-brane $\mathcal{B}_{(x,\alpha)}$ and a D1-brane $V^{(1)}_\beta$, we need to take the flat line bundle on the bi-brane into account. We first pull back the line bundle on $V^{(1)}_\beta$ along the projection p_2 to a line bundle on $S^1_R \times S^1_R$; then we restrict it to the world volume B_x of $\mathcal{B}_{(x,\alpha)}$ and tensor this restriction with the line bundle on $\mathcal{B}_{(x,\alpha)}$ described by the Wilson line α. This gives a line bundle with Wilson line $\alpha + \beta$ on the world volume of the bi-brane that can be pushed down along the projection p_1 to a line bundle with the same Wilson line on S^1_R. We conclude that

$$\mathcal{B}_{(x,\alpha)} \star V^{(1)}_\beta = V^{(1)}_{\alpha+\beta}. \tag{53}$$

Thus the fusion with a defect of type $\mathcal{B}_{(x,\alpha)}$ acts on D1-branes as a translation by α in the space of Wilson lines.

We can similarly compute the fusion of two bi-branes $\mathcal{B}_{(x,\alpha)}$ and $\mathcal{B}_{(x',\alpha')}$: we have

$$\begin{aligned} &p_{12}^{-1}(B_x) = \{(y, y - x, y') \mid y, y' \in [0, 2\pi R)\}, \\ &p_{23}^{-1}(B_{x'}) = \{(y, y', y' - x') \mid y, y' \in [0, 2\pi R)\}, \\ &p_{13}\left(p_{12}^{-1}(B_x) \cap p_{23}^{-1}(B_{x'})\right) = \{(y, y - x - x') \mid y \in [0, 2\pi R)\}, \end{aligned} \tag{54}$$

so that the position variables of bi-branes add up under fusion. To understand the behaviour of Wilson lines, we take into account the flat line bundles by pulling them back to $S^1_R \times S^1_R \times S^1_R$ and tensoring them. Then as in the case of mixed fusion, the Wilson lines add up. We thus obtain

$$\mathcal{B}_{(x_1,\alpha_1)} \star \mathcal{B}_{(x_2,\alpha_2)} = \mathcal{B}_{(x_1+x_2,\alpha_1+\alpha_2)}. \tag{55}$$

Hence we find that both the position and Wilson line variable of bi-branes add up under fusion. This result exactly matches the fusion of the first set of defects that are derived algebraically in [16]; for these both the left- and right-moving currents are preserved, $J_1(z) = J_2(z)$ and $\bar{J}_1(\bar{z}) = \bar{J}_2(\bar{z})$, for z a point on the defect line. One can also consider the case where one or both of the currents are only preserved up to a non-trivial automorphism; the $\mathfrak{u}(1)$ current algebra has only a single non-trivial automorphism, acting as $J \mapsto -J$. The simplest case then turns out to be that both $J_1(z) = -J_2(z)$ and $\bar{J}_1(\bar{z}) = -\bar{J}_2(\bar{z})$; in this case one obtains submanifolds of the form $B = \{(y \bmod 2\pi R\mathbb{Z}, h - y \bmod 2\pi R\mathbb{Z}) | y \in \mathbb{R}\}$. The case of different automorphisms for left movers and right movers is more subtle; we expect the corresponding bi-branes to fill the whole product space. Also, formula (37) suggests that the 2-form on the bi-brane should be proportional to $\pm\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2$, with the sign depending on the chirality on which the non-trivial automorphism acts. These issues will not be addressed in the present paper.

5.3. *Bi-branes for the compactified free boson at different radii*

We next turn our attention to bi-branes which describe topological defects that separate a region which supports a boson compactified on a circle of radius R_1 from a region supporting a boson compactified at radius R_2. We describe the product space by two coordinates x_1 and x_2, with x_i to be taken modulo $2\pi R_i\mathbb{Z}$. The bi-brane world volumes are

$$B_h := \{(y \bmod 2\pi R_1\mathbb{Z},\, y - h \bmod 2\pi R_2\mathbb{Z}) \mid y \in \mathbb{R}\}. \tag{56}$$

If the ratio R_1/R_2 is not rational, this set is isomorphic to $\mathbb{R}$ and fills $S^1_{R_1} \times S^1_{R_2}$ densely. Accordingly there are no Wilson line variables. The algebraic approach shows that in this situation there is a single defect that preserves all current symmetries [16]; in particular, h is not a physical parameter.

We thus assume that the ratio of the two radii is rational,

$$R_1/R_2 = r/s \tag{57}$$

with r, s coprime positive integers. The bi-brane world volume then has length $2\pi sR_1 = 2\pi rR_2$ and admits a Wilson line variable, to be taken modulo $1/(2\pi sR_1) = 1/(2\pi rR_2)$. It wraps s times in the R_1-direction; hence the geometric parameter, when measured on the x_2-axis, is reduced to $2\pi R_2/s$. Equivalently, it wraps r times in the R_1-direction; hence the geometric parameter, if measured on the x_1-axis, is reduced to $2\pi R_1/r$. Thus the position parameter is to be taken modulo $2\pi R_1/r = 2\pi R_2/s$.

This should again be compared to the analysis of [16]. In the case at hand two parameters have been found: the first couples to the sum of left- and right-moving momenta, which by the compatibility of the two radii is required to be quantized in units of r/R_1. This nicely fits the position parameter found above. Similarly, there is a parameter coupling to winding, i.e. to the difference of left- and right-moving momenta. The latter is quantized in units of sR_1, fitting the quantization of the Wilson lines derived above.

Again one can generalize the analysis to bi-branes that preserve the chiral currents only up to automorphisms. If the non-trivial automorphism is taken for both chiralities, one expects off-diagonal bi-branes; the discussion of the parameters largely parallels the one in the preceding paragraphs. In the case of different automorphisms, one expects bi-branes filling $S^1_{R_1} \times S^1_{R_2}$, provided that the area of the product space is rational in suitable units. For the specific case $R_2 = 2/R_1$ these bi-branes should be related to defects which implement T-duality. In this context, the fact [23] that the curvature $\pm\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2$ is of the same form as the curvature of the Poincaré line bundle is highly intriguing. A careful discussion of this relationship is, again, beyond the scope of the present paper.

5.4. *WZW bi-branes*

We now turn our attention to bi-branes of WZW models on simply connected compact Lie groups. Here several new phenomena arise: the position of possible branes and bi-branes is quantized, and multiplicities other than zero or one are expected from the algebraic approach. In fact, from that approach it is known that for these theories the multiplicities appearing in the fusion of bi-branes as well as the mixed fusion of bi-branes and branes are the same as the chiral fusion multiplicities which are given by the Verlinde formula.

To analyze this issue, it turns out to be convenient to work with fusion coefficients of type $\mathcal{N}_{\alpha\beta\gamma}$; here α and γ are group elements characterizing conjugacy classes $\mathcal{C}_\alpha$ and $\mathcal{C}_\gamma$ of G, respectively, which support a brane, while β is a group element characterizing a bi-brane $\tilde{\mu}^{-1}(\mathcal{C}_\beta)$ with $\tilde{\mu}$ as in (24). In the sequel we assume that all group elements are regular, i.e. contained in just a single maximal torus of G. We are thus led to consider the subset

$$\begin{aligned} M_{\alpha\beta\gamma} &:= p_1^{-1}(\mathcal{C}_\alpha) \cap \tilde{\mu}^{-1}(\mathcal{C}_\beta) \cap p_2^{-1}(\mathcal{C}_\gamma) \\ &= \{(g_1, g_2) \in G \times G \mid g_1 \in \mathcal{C}_\alpha, g_2 \in \mathcal{C}_\gamma, g_1 g_2^{-1} \in \mathcal{C}_\beta\} \end{aligned} \tag{58}$$

of $G \times G$. This set is equipped with a natural G-action, obtained by combining the adjoint action on g_1 and on g_2. Both branes and bi-branes are equipped with 2-forms; as a consequence, $M_{\alpha\beta\gamma}$ comes with a natural 2-form, namely the sum

$$\omega_{\alpha\beta\gamma} := p_1^*\omega_\alpha\big|_{M_{\alpha\beta\gamma}} + p_2^*\omega_\gamma\big|_{M_{\alpha\beta\gamma}} + \varpi_\beta\big|_{M_{\alpha\beta\gamma}} \tag{59}$$

of the restrictions of the three 2-forms $p_1^*\omega_\alpha$, $p_2^*\omega_\gamma$ and ϖ_β.

According to the results obtained in the algebraic approach, this space should be linked to the fusion rules of the chiral WZW theory at level k. To see how such a relation can exist, we recall that fusion rules are dimensions of spaces of conformal blocks. The latter can be obtained by geometric quantization from suitable moduli spaces of flat connections; as such they arise in the quantization of Chern–Simons theories.

The situation relevant for Verlinde multiplicities is given by the three-punctured sphere $S^2_{(3)}$, also known as the ‘pair of pants’ or trinion. In classical Chern–Simons theory one considers the moduli space of flat connections on S^2 whose monodromy around the three insertion points takes values in conjugacy classes $\mathcal{C}_\alpha$, $\mathcal{C}_\beta$ and $\mathcal{C}_\gamma$, respectively. Taking the monodromies $g_\alpha \in \mathcal{C}_\alpha$, $g_\beta \in \mathcal{C}_\beta$ and $g_\gamma \in \mathcal{C}_\gamma$ along circles of the same orientation around all three insertions, the relations in the fundamental group of the trinion impose that $g_\alpha g_\beta g_\gamma = 1$. Since monodromies are defined only up to simultaneous conjugation, the moduli space that matters in classical Chern–Simons theory is isomorphic to the quotient $M_{\alpha\beta\gamma}/G$.

Note that the bounds on the range of bi-branes that appear in the fusion are already present before geometric quantization. Indeed, the relevant product

$$\mathcal{C}_h * \mathcal{C}_{h'} := \{gg' | g \in \mathcal{C}_h, g' \in \mathcal{C}_{h'}\} \tag{60}$$

of conjugacy classes has already been considered, for $G = SU(2)$, in [24]. It is convenient to characterize a conjugacy class of $SU(2)$ by its trace or, equivalently, by the angle θ with

$$\cos\theta = \frac{1}{2}\mathrm{tr}(g), \tag{61}$$

which takes values $\theta \in [0, \pi]$. One finds (see Proposition 3.1 of [24]) that the (classical) product (60) of the two conjugacy classes with angles θ, θ' is the union of all conjugacy classes with angle θ'' in the range

$$|\theta - \theta'| \le \theta'' \le \min\{\theta + \theta', 2\pi - (\theta + \theta')\}. \tag{62}$$

This already yields the correct upper and lower bounds that appear in the $SU(2)$ fusion rules.

A full understanding of fusion can only be expected after applying geometric quantization to the so obtained moduli space: this space must be endowed with a 2-form, which is interpreted as the curvature of a line bundle, and the holomorphic sections of this bundle are what results from geometric quantization. In view of this need for quantization it is a highly non-trivial observation that the 2-form (59) furnished by the two branes and the bi-brane is exactly the same as the one which arises[3] from classical Chern–Simons theory.

[3] We are grateful to Anton Alekseev for information about this 2-form.

6. Outlook

Our findings naturally admit various extensions and generalizations. For instance, one can impose conservation of the currents only up to an automorphism of the horizontal Lie algebra, which may be chosen independently for left- and right-moving degrees of freedom. Also, our methods can be clearly extended to more general classes of conformal field theories, in particular to WZW models on non-simply connected groups, coset models, as well as to theories of several free bosons compactified on a torus and to orbifolds thereof, including asymmetric orbifolds such as lens spaces. Another generalization concerns defects which separate sigma models on two different Lie groups that share the same Lie algebra.

Furthermore, our results provide independent evidence for the idea that there is an intimate relation between defects and correspondences. This idea has played a role in a field theoretic realization of the geometric Langlands program (see Section 6.4 of [25]). It is therefore not unreasonable to expect that defects and, more generally, the algebraic and categorical structure behind RCFT correlators, will enter in a CFT-inspired approach to the Langlands program.

Finally it could be rewarding to unravel similar structures in lattice models.

Acknowledgements

We thank Birgit Richter for a helpful discussion and Anton Alekseev for a helpful correspondence. J.F. is partially supported by VR under project no. 621-2006-3343. C.S. and K.W. are supported by the Sonderforschungsbereich “Particles, Strings and the Early Universe — the Structure of Matter and Space–Time”.

Appendix A. Birelative (co)homology

In this Appendix we discuss the well-definedness of the Wess–Zumino term (45) in the presence of a defect line. To this end we set up a homology theory based on singular homology, which can be understood as a generalization of relative homology, and which we will accordingly call birelative homology. The associated cohomology theory with real coefficients can be identified with a cohomology theory based on differential forms, which we call birelative de Rham cohomology. These structures enable us to formulate precise conditions under which the Wess–Zumino term (45) is well-defined up to integers.

Recall that the (singular) homology $H_k(M)$ of a smooth manifold M is the homology of the singular chain complex with chain groups $\Delta_k(M)$, consisting of (smooth) k-simplices in M and boundary operator $\partial : \Delta_k(M) \to \Delta_{k-1}(M)$ (we suppress the index of the boundary operator ∂, as it can be inferred from the index of the simplex on which it acts). If $Q \subset M_1 \times M_2$ is a submanifold, we define the kth *birelative chain group* of the triple (M_1, M_2, Q) to be

$$\Delta_k(M_1, M_2, Q) := \Delta_k(M_1) \oplus \Delta_k(M_2) \oplus \Delta_{k-1}(Q). \tag{A.1}$$

Using the projections $p_i : M_1 \times M_2 \to M_i$ and the inclusion map $\iota : Q \hookrightarrow M_1 \times M_2$, and the induced chain maps $(p_i)_*$ and ι_*, we define the homomorphism

$$\begin{aligned} &\partial : \Delta_k(M_1, M_2, Q) \to \Delta_{k-1}(M_1, M_2, Q) \\ &(\sigma_1, \sigma_2, \tau) \mapsto (\partial\sigma_1 + (p_1)_*\iota_*\tau, \partial\sigma_2 - (p_2)_*\iota_*\tau, -\partial\tau). \end{aligned} \tag{A.2}$$

It is easy to verify that this map satisfies $\partial^2 = 0$, i.e. we have endowed the birelative chain groups with the structure of a complex. We call its homology groups the *birelative homology groups* and denote them by $H_k(M_1, M_2, Q)$. Explicitly, an element of $H_k(M_1, M_2, Q)$ is represented by a triple $(\sigma_1, \sigma_2, \tau)$ of chains $\sigma_i \in \Delta_k(M_i)$, $i = 1, 2$, and a cycle $\tau \in \Delta_{k-1}(Q)$, such that $\partial\sigma_1 = (p_1)_*\iota_*\tau$ and $\partial\sigma_2 = -(p_2)_*\iota_*\tau$. For each degree k, the birelative chain group fits, by definition, into the short exact sequence

$$0 \longrightarrow \Delta_k(M_1) \oplus \Delta_k(M_2) \xrightarrow{\alpha} \Delta_k(M_1, M_2, Q) \xrightarrow{\beta} \Delta_{k-1}(Q) \longrightarrow 0, \tag{A.3}$$

in which α is the inclusion and β is the projection. These induce a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_k(M_1) \oplus H_k(M_2) \arrow[r, "\omega"] & H_k(M_1, M_2, Q) \arrow[dll, rounded corners] \\ & H_{k-1}(Q) \arrow[r] & H_{k-1}(M_1) \oplus H_{k-1}(M_2) \arrow[r] & \cdots \end{tikzcd} \tag{A.4}$$

in homology.

To explain the term birelative homology we observe that we have generalized relative homology in the following sense: if we take $M_2 = pt$, so that we can identify Q with a submanifold of M_1, then there is a canonical isomorphism $H_k(M_1, pt, Q) \to H_k(M_1, Q)$. Here $H_k(M_1, Q)$, the relative homology group of M_1 with respect to the submanifold Q, is constructed as the homomorphism $[(\sigma_1, \sigma_2, \tau)] \mapsto [\sigma_1]$ which can be shown to be an isomorphism by using the 5-lemma (see e.g. [5], Lemma IV.5.10) applied to the exact sequence (A.4) and the corresponding sequence in relative homology.

Dual to the singular homology groups there are singular cohomology groups, defined to be the cohomology of a complex whose cochain groups are

$$\Delta^k(M, R) := \mathrm{Hom}(\Delta_k(M), R) \tag{A.5}$$

for a coefficient ring R, and whose coboundary operator

$$\delta : \Delta^k(M, R) \to \Delta^{k+1}(M, R) \tag{A.6}$$

is given by $\delta\varphi(\sigma) := \varphi(\partial\sigma)$ for any $(k+1)$-simplex σ in M. There is a canonical pairing

$$H^k(M, R) \times H_k(M) \to R \quad \text{with } ([\varphi], [\sigma]) \mapsto \varphi(\sigma), \tag{A.7}$$

which is easily seen to be well-defined. It is often convenient to recover the cohomology groups with values in the real numbers in a geometric way, for instance through differential forms. Let us recall how this works. The integrals of k-forms $\varphi \in \Omega^k(M)$ over k-simplices $\sigma \in \Delta_k(M)$ define homomorphisms $\Psi_k : \Omega^k(M) \to \Delta^k(M, \mathbb{R})$ which, by Stokes' theorem, fit together to a chain map. The induced homomorphism

$$\Psi^* : H^k_{\mathrm{dR}}(M) \to H^k(M, \mathbb{R}) \tag{A.8}$$

from de Rham cohomology to singular cohomology is an isomorphism, which is known as the de Rham isomorphism (see e.g. Theorem V.9.1 of [5]).

Analogously, as for ordinary singular cohomology, we can also define birelative cohomology. Thus there are birelative cochain groups $\Delta^k(M_1, M_2, Q, R)$, birelative cohomology groups $H^k(M_1, M_2, Q, R)$, and a canonical pairing

$$H^k(M_1, M_2, Q, R) \times H_k(M_1, M_2, Q) \to R. \tag{A.9}$$

Note that because the exact sequence (A.3) splits, the dual sequence

$$0 \longrightarrow \Delta^{k-1}(Q, R) \longrightarrow \Delta^k(M_1, M_2, R) \longrightarrow \Delta^k(M_1) \oplus \Delta^k(M_2) \longrightarrow 0 \tag{A.10}$$

is exact, too, and induces a long exact sequence in cohomology. We would like be able to express the birelative cohomology groups with real coefficients by differential forms in a similar way to how the de Rham isomorphism does it for ordinary cohomology. To this end we consider the vector spaces

$$\Omega^k(M_1, M_2, Q) := \Omega^k(M_1) \oplus \Omega^k(M_2) \oplus \Omega^{k-1}(Q) \tag{A.11}$$

together with the linear maps

$$\begin{aligned} &\mathrm{d} : \Omega^k(M_1, M_2, Q) \to \Omega^{k+1}(M_1, M_2, Q) \\ &(H_1, H_2, \varpi) \mapsto (\mathrm{d}H_1, \mathrm{d}H_2, \iota^*(p_1^* H_1 - p_2^* H_2) - \mathrm{d}\varpi). \end{aligned} \tag{A.12}$$

This indeed defines a complex:

$$\begin{aligned} \mathrm{d}^2(H_1, H_2, \varpi) &= \mathrm{d}(\mathrm{d}H_1, \mathrm{d}H_2, \iota^*(p_1^* H_1 - p_2^* H_2) - \mathrm{d}\varpi) \\ &= (\mathrm{d}^2 H_1, \mathrm{d}^2 H_2, \iota^*(p_1^* \mathrm{d}H_1 - p_2^* \mathrm{d}H_2) - \mathrm{d}\iota^*(p_1^* H_1 - p_2^* H_2) + \mathrm{d}^2\varpi) \\ &= (0, 0, 0). \end{aligned} \tag{A.13}$$

We call the cohomology of this complex the *birelative de Rham cohomology* and denote it by $H^k_{\mathrm{dR}}(M_1, M_2, Q)$. By putting $M_2 = pt$, this is nothing but the relative de Rham cohomology of the map $\iota : Q \to M$; see e.g. I Section 6 of [4].

Notice that a simply connected M_1–M_2-bi-brane (Q, ϖ) provides us with an element (H_1, H_2, ϖ) of $\Omega^3(M_1, M_2, Q)$. The condition (41) on the 2-form ϖ on the bi-brane shows that (H_1, H_2, ϖ) is closed and thus defines a class in the birelative de Rham cohomology.

Like the definition of the homomorphism $\Psi : \Omega^k(M) \to \Delta^k(M, \mathbb{R})$ mentioned above we obtain a natural homomorphism

$$\Psi_{\mathrm{bi}} : \Omega^k(M_1, M_2, Q) \to \Delta^k(M_1, M_2, Q, \mathbb{R}) \tag{A.14}$$

which by definition associates with a triple $(H_1, H_2, \varpi) \in \Omega^k(M_1, M_2, Q)$ evaluated on an element $(\sigma_1, \sigma_2, \tau) \in \Delta_k(M_1, M_2, Q)$ the real number

$$\Psi_{\mathrm{bi}}(H_1, H_2, \varpi)(\sigma_1, \sigma_2, \tau) := \int_{\sigma_1} H_1 + \int_{\sigma_2} H_2 + \int_{\tau} \varpi. \tag{A.15}$$

The homomorphisms Ψ_{bi} fit together to a chain map:

$$\begin{aligned} (\delta\Psi_{\mathrm{bi}}(H_1, H_2, \varpi))(\sigma_1, \sigma_2, \tau) &= \Psi_{\mathrm{bi}}(H_1, H_2, \varpi)(\partial\sigma_1 + (p_1)_*\iota_*\tau, \partial\sigma_2 - (p_2)_*\iota_*\tau, -\partial\tau) \\ &= \int_{\partial\sigma_1 + (p_1)_*\iota_*\tau} H_1 + \int_{\partial\sigma_2 - (p_2)_*\iota_*\tau} H_2 + \int_{-\partial\tau} \varpi \\ &= \int_{\sigma_1} \mathrm{d}H_1 + \int_{\sigma_2} \mathrm{d}H_2 + \int_{\tau} \iota^*(p_1^* H_1 - p_2^* H_2) - \mathrm{d}\varpi \\ &= \Psi_{\mathrm{bi}}(\mathrm{d}H_1, \mathrm{d}H_2, \iota^*(p_1^* H_1 - p_2^* H_2) - \mathrm{d}\varpi)(\sigma_1, \sigma_2, \tau) \\ &= \Psi_{\mathrm{bi}}(\mathrm{d}(H_1, H_2, \varpi))(\sigma_1, \sigma_2, \tau). \end{aligned} \tag{A.16}$$

We infer that the induced homomorphism

$$\Psi^*_{\mathrm{bi}} : H^k_{\mathrm{dR}}(M_1, M_2, Q) \to H^k(M_1, M_2, Q, \mathbb{R}) \tag{A.17}$$

is an isomorphism, analogously with the de Rham isomorphism. To prove this claim, note that by definition we have an exact sequence

$$0 \longrightarrow \Omega^{k-1}(Q) \xrightarrow{\alpha} \Omega^k(M_1, M_2, Q) \xrightarrow{\beta} \Omega^k(M_1) \oplus \Omega^k(M_2) \longrightarrow 0, \tag{A.18}$$

where $\alpha(\varpi) := (0, 0, \varpi)$ and $\beta(H_1, H_2, \varpi) := (H_1, H_2)$. It induces a long exact sequence

$$\begin{array}{l} \cdots \longrightarrow H^{k-1}_{\mathrm{dR}}(Q) \xrightarrow{\alpha^*} H^k_{\mathrm{dR}}(M_1, M_2, Q) \xrightarrow{\beta^*} H^k_{\mathrm{dR}}(M_1) \oplus H^k_{\mathrm{dR}}(M_2) \\ \longrightarrow H^k_{\mathrm{dR}}(Q) \longrightarrow \cdots \end{array} \tag{A.19}$$

in (birelative) de Rham cohomology. Combining with the long exact sequence in birelative cohomology with values in $\mathbb{R}$, induced by the exact sequence (A.10), we have the following diagram with exact columns:

$$\begin{tikzcd}
H^{k-1}_{\mathrm{dR}}(M_1) \oplus H^{k-1}_{\mathrm{dR}}(M_2) \arrow[r, "\Psi^* \oplus \Psi^*"] \arrow[d] & H^{k-1}(M_1, \mathbb{R}) \oplus H^{k-1}(M_2, \mathbb{R}) \arrow[d] \\
H^{k-1}_{\mathrm{dR}}(Q) \arrow[r, "\Psi^*"] \arrow[d] & H^{k-1}(Q, \mathbb{R}) \arrow[d] \\
H^{k}_{\mathrm{dR}}(M_1, M_2, Q) \arrow[r, "\Psi^*_{\mathrm{bi}}"] \arrow[d] & H^{k}(M_1, M_2, Q, \mathbb{R}) \arrow[d] \\
H^{k}_{\mathrm{dR}}(M_1) \oplus H^{k}_{\mathrm{dR}}(M_2) \arrow[r, "\Psi^* \oplus \Psi^*"] \arrow[d] & H^{k}(M_1, \mathbb{R}) \oplus H^{k}(M_2, \mathbb{R}) \arrow[d] \\
H^{k}_{\mathrm{dR}}(Q) \arrow[r, "\Psi^*"'] & H^{k}(Q, \mathbb{R})
\end{tikzcd} \tag{A.20}$$

It is easy to check that all subdiagrams commute, so that the 5-lemma implies that Ψ^*_{bi} is an isomorphism.

In the same way as for ordinary cohomology, we say that a cocycle in $\Omega^k(M_1, M_2, Q)$ is integral iff its class – identified by Ψ^*_{bi} with a class in $H^k(M_1, M_2, Q, \mathbb{R})$ – lies in the image of the induced homomorphism

$$H^k(M_1, M_2, Q, \mathbb{Z}) \to H^k(M_1, M_2, Q, \mathbb{R}). \tag{A.21}$$

In this case the canonical pairing (A.9) of $\Psi^*_{\mathrm{bi}}([H_1, H_2, \varpi])$ with any birelative homology class $[(\sigma_1, \sigma_2, \tau)]$, which is given by

$$\int_{\sigma_1} H_1 + \int_{\sigma_2} H_2 + \int_{\tau} \varpi, \tag{A.22}$$

is an integer. Analogously, as for WZW models in the bulk and on the boundary of a world sheet, this notion of integral classes is essential for achieving the well-definedness of Wess–Zumino terms. We infer the following result:

The Wess–Zumino term $S[\phi_1, \phi_2]$ (45) of a simply connected M_1–M_2-bi-brane (Q, ϖ) is well-defined up to integers, provided that the class of (H_1, H_2, ϖ) in the birelative de Rham cohomology group $H^3_{\mathrm{dR}}(M_1, M_2, Q)$ is integral.

To prove this claim, recall that the definition of $S[\phi_1, \phi_2]$ involves choices of submanifolds D of Q and B_i of M_i. If we represent these submanifolds as singular chains, then

$$\partial D = \phi_S(S), \qquad \partial B_1 = \phi_1(\Sigma_1) - (p_1)_* D \quad \text{and} \quad \partial B_2 = \phi_2(\Sigma_2) + (p_2)_* D. \tag{A.23}$$

Consider now different choices D', B'_1 and B'_2, and let $\tau := D - D'$ be a chain in $\Delta_2(Q)$ and $\sigma_i := B_i - B'_i$ be chains in $\Delta_3(M_i)$. We find

$$\partial \tau = 0, \qquad \partial \sigma_1 = -(p_1)_* \tau \quad \text{and} \quad \partial \sigma_2 = (p_2)_* \tau, \tag{A.24}$$

so that $(\sigma_1, \sigma_2, \tau)$ defines a class in the birelative homology $H_3(M_1, M_2, Q)$. The ambiguities of the Wess–Zumino term $S[\phi_1, \phi_2]$ are thus of the form

$$\left(\int_{B_1} H_1 + \int_{B_2} H_2 + \int_D \varpi \right) - \left(\int_{B'_1} H_1 + \int_{B'_2} H_2 + \int_{D'} \varpi \right) = \int_{\sigma_1} H_1 + \int_{\sigma_2} H_2 + \int_{\tau} \varpi. \tag{A.25}$$

In view of (A.15) the ambiguities (A.25) are nothing but the pairing of the cycle $(\sigma_1, \sigma_2, \tau)$ with (H_1, H_2, ϖ). If (H_1, H_2, ϖ) is integral, this gives an integer.

Appendix B. Bundle gerbes and defects

As we have explained in Section 4 it is perfectly accurate to characterize bundle gerbes on 2-connected target spaces M_1 and M_2 by their curvature 3-forms H_1 and H_2. Under this condition, we have defined an M_1–M_2-bi-brane to be a simply connected submanifold Q of $M_1 \times M_2$ together with a 2-form ϖ on Q that obeys

$$p_1^* H|_Q = p_2^* H|_Q + \mathrm{d}\varpi. \tag{B.1}$$

In this appendix we generalize this definition to bi-branes between target spaces with are not 2-connected. This makes it necessary to work with the full structure of a hermitian bundle gerbe. Examples of non-2-connected target spaces are provided by non-simply connected Lie groups, such as the group $SO(4n)/\mathbb{Z}_2$, which admits two non-isomorphic bundle gerbes with the same curvature 3-form H. At the same time, we drop the restriction on the bi-brane Q of being simply connected. Examples of non-simply connected bi-branes are provided by certain biconjugacy classes of non-simply connected Lie groups.

B.1. Gerbe modules

Let us first recall how branes have been understood using bundle gerbes [21,20]. Let $\mathcal{G}$ be a bundle gerbe on the target space M with curvature H. The geometric structure related to a conformal boundary condition consists of a pair[4] $(Q, \mathcal{E})$, with Q a submanifold of M and $\mathcal{E}$ a gerbe module for the restriction of $\mathcal{G}$ to Q. Such gerbe modules are vector bundles twisted by the bundle gerbe $\mathcal{G}$. We can view them as bundle gerbe morphisms

$$\mathcal{E} : \mathcal{G}|_Q \to \mathcal{I}_\omega \tag{B.2}$$

from $\mathcal{G}|_Q$ to a trivial bundle gerbe $\mathcal{I}_\omega$ given by a 2-form ω on Q [28]. The 2-form ω is called the curvature of the gerbe module. A necessary condition for the existence of the morphism $\mathcal{E}$ is the equality

$$H|_Q = \mathrm{d}\omega \tag{B.3}$$

on Q. If the submanifold Q is not simply connected, then non-trivial flat line bundles exist. Since gerbe modules (of equal rank) with the same curvature ω form a torsor over the group of flat line bundles, in this situation non-isomorphic gerbe modules with the same curvature exist. This happens, for example, for the equatorial conjugacy class of $SO(3)$, which has the topology of $\mathbb{RP}^2$ and thus admits two non-isomorphic flat line bundles, whose action relates two non-isomorphic gerbe modules.

The arguably most direct way to understand (hermitian) bundle gerbes (with connective structure) is in terms of their local data: with respect to a good open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of M, a bundle gerbe $\mathcal{G}$ can be described by a collection (g_{ijk}, A_{ij}, B_i) of smooth functions $g_{ijk} : U_i \cap U_j \cap U_k \to U(1)$, 1-forms $A_{ij} \in \Omega^1(U_i \cap U_j)$ and 2-forms $B_i \in \Omega^2(U_i)$ satisfying the cocycle conditions

$$\begin{aligned} &g_{jkl}^{-1} \cdot g_{ikl} \cdot g_{ijl}^{-1} \cdot g_{ijk} = 1 \quad \text{on } U_i \cap U_j \cap U_k \cap U_l, \\ &-\mathrm{i}g_{ijk}^{-1}\mathrm{d}g_{ijk} + A_{jk} - A_{ik} + A_{ij} = 0 \quad \text{on } U_i \cap U_j \cap U_k, \\ &\mathrm{d}A_{ij} - B_j + B_i = 0 \quad \text{on } U_i \cap U_j. \end{aligned} \tag{B.4}$$

The curvature of $\mathcal{G}$ is the globally defined 3-form H with $H|_{U_i} := \mathrm{d}B_i$. For example, the local data of the trivial bundle gerbe $\mathcal{I}_\omega$ are $(1, 0, \omega|_{U_i\cap Q})$. A rank-$n$ bundle gerbe module $\mathcal{E} : \mathcal{G}|_Q \to \mathcal{I}_\omega$ is in this formalism described by a collection (G_{ij}, Π_i) of smooth functions $G_{ij} : U_i \cap U_j \cap Q \to U(n)$ and $\mathfrak{u}(n)$-valued 1-forms $\Pi_i \in \Omega^1(U_i \cap Q) \otimes \mathfrak{u}(n)$ which relate the local data of the bundle gerbes $\mathcal{G}|_Q$ and $\mathcal{I}_\omega$ in the following way:

$$\begin{aligned} &1 = g_{ijk} \cdot G_{ik} G_{jk}^{-1} G_{ij}^{-1} \quad \text{on } Q \cap U_i \cap U_j \cap U_k, \\ &0 = A_{ij} + \Pi_j - G_{ij}^{-1} \Pi_i G_{ij} - \mathrm{i}G_{ij}^{-1}\mathrm{d}G_{ij} \quad \text{on } Q \cap U_i \cap U_j, \\ &\omega = B_i + \frac{1}{n}\mathrm{tr}(\mathrm{d}\Pi_i) \quad \text{on } Q \cap U_i. \end{aligned} \tag{B.5}$$

[4] But not every such pair corresponds to a conformal boundary condition; there are far more such pairs than conformal boundary conditions.

Note that the derivative of the last equality reproduces the relation (B.3). Also note that if the bundle gerbe $\mathcal{G}$ is itself trivial, i.e. has local data $(1, 0, B|_{U_i})$ for a globally defined Kalb–Ramond field $B \in \Omega^2(M)$, then (G_{ij}, Π_i) are the local data of a rank-n vector bundle over Q with curvature of trace $n(\omega - B)$. This explains the terminology "twisted" vector bundle in the non-trivial case. Finally, notice that if one changes (G_{ij}, Π_i) with local data of a non-trivializable flat vector bundle over the world volume Q of the bi-brane, then one obtains a new bundle gerbe module with the same curvature. In this way the existence of non-trivial flat vector bundles over Q makes the use of bundle gerbe modules unavoidable.

In the case of WZW conformal field theories with $M = G$ one considers in particular so-called symmetric branes, which preserve the current algebra in the presence of boundaries, and thus in particular conformal invariance. Symmetric D-branes $(Q, \mathcal{E})$ can be characterized by three conditions [20]:

(1) the world volume Q of the brane is a conjugacy class $\mathcal{C}_h$ of G;
(2) the local 2-forms $\mathrm{d}\Pi_i$ take their values only in the center of the Lie algebra $\mathfrak{u}(n)$ and can thus be identified with real 2-forms;
(3) the 2-form ω is fixed to

$$\omega = \left\langle \theta|_{\mathcal{C}_h} \wedge \frac{\mathrm{Ad}^{-1}+1}{\mathrm{Ad}^{-1}-1}\,\theta|_{\mathcal{C}_h} \right\rangle. \tag{B.6}$$

The conditions 2 and 3 restrict the choice of the conjugacy class to conjugacy classes that correspond to integrable highest weights. This amounts in particular to having a finite number of non-intersecting brane world volumes.

B.2. Gerbe bimodules

That bundle gerbe modules are the appropriate structure for branes in the case of non-2-connected target spaces or non-simply connected supports, together with the folding trick, suggests the corresponding structure as the appropriate generalization for bi-branes: for bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ over M_1 and M_2, an *M_1–M_2-bi-brane* is a submanifold $Q \subset M_1 \times M_2$ together with a $(p_1^*\mathcal{G}_1)|_Q$-$(p_2^*\mathcal{G}_2)|_Q$-bimodule: a bundle gerbe morphism

$$\mathcal{D} : (p_1^*\mathcal{G}_1)\big|_Q \to (p_2^*\mathcal{G}_2)\big|_Q \otimes \mathcal{I}_{\varpi} \tag{B.7}$$

with ϖ as in (B.1). Here we shall call the 2-form ϖ the curvature of the bimodule. This definition is related to the folding trick in the sense, that – using the appropriate notion of duality for bundle gerbes (see Section 1.4 of [28]) – a $\mathcal{G}_1$–$\mathcal{G}_2$-bimodule is the same as a $(\mathcal{G}_1 \otimes \mathcal{G}_2^*)$-module.

To consider a bundle gerbe bimodule $\mathcal{D}$ in the local data formalism, let $\mathfrak{U}$ be a good covering of $M_1 \times M_2$, let (g_{ijk}, A_{ij}, B_i) be local data of $p_1^*\mathcal{G}_1$, and $(g'_{ijk}, A'_{ij}, B'_i)$ local data of $p_2^*\mathcal{G}_2$. Then the bimodule has local data (G_{ij}, Π_i) like for a bundle gerbe module, but now satisfying

$$\begin{aligned} g'_{ijk} &= g_{ijk} \cdot G_{ik} G_{jk}^{-1} G_{ij}^{-1} \quad \text{on } Q \cap U_i \cap U_j \cap U_k, \\ A'_{ij} &= A_{ij} + \Pi_j - G_{ij}^{-1} \Pi_i G_{ij} - \mathrm{i} G_{ij}^{-1} \mathrm{d} G_{ij} \quad \text{on } Q \cap U_i \cap U_j, \\ B'_i + \varpi &= B_i + \frac{1}{n} \mathrm{tr}(\mathrm{d}\Pi_i) \quad \text{on } Q \cap U_i. \end{aligned} \tag{B.8}$$

Again we make three observations. First, the derivative of the third equality gives Eq. (B.1); second, if both bundle gerbes $p_1^*\mathcal{G}_1$ and $p_2^*\mathcal{G}_2$ are trivial, then a bimodule is just a rank-n vector bundle over Q with curvature of trace $n(B' - B + \varpi)$; and third, we can still change the local data (G_{ij}, Π_i) with local data of a flat vector bundle over Q and obtain another bimodule with the same curvature. Such phenomena arise, in particular, for bi-branes for WZW theories on non-simply connected Lie groups.

B.3. Holonomy in the presence of defects

We have generalized the definition of bi-branes from simply connected bi-branes between 2-connected target spaces with 3-forms to arbitrary bi-branes between arbitrary target spaces with bundle gerbes. Now we shall generalize the Wess–Zumino term for bi-branes as given in (45) to the general case as well.

Let M_1 and M_2 be smooth manifolds with bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, and let $(Q, \mathcal{E})$ be a bi-brane, i.e. a submanifold Q of $M_1 \times M_2$ together with a $(p_1^*\mathcal{G}_1)|_Q$-$(p_2^*\mathcal{G}_2)|_Q$-bimodule

$$\mathcal{D} : (p_1^*\mathcal{G}_1)\big|_Q \to (p_2^*\mathcal{G}_2)\big|_Q \otimes \mathcal{I}_\varpi \tag{B.9}$$

with curvature ϖ. Recall that we defined the Wess–Zumino term for the following situation: a closed oriented world sheet Σ with an embedded oriented circle $S \subset \Sigma$, which separates the world sheet into two components, $\Sigma = \Sigma_1 \cup_S \Sigma_2$, together with maps $\phi_i : \Sigma_i \to M_i$ for $i = 1, 2$ such that the image of the combined map

$$\begin{aligned} \phi_S : \quad & S \to M_1 \times M_2 \\ & s \mapsto (\phi_1(s), \phi_2(s)) \end{aligned} \tag{B.10}$$

is contained in Q. The orientation of Σ_i is the one inherited from the orientation of Σ, and without loss of generality we take $\partial\Sigma_1 = S$ and $\partial\Sigma_2 = \overline{S}$.

To define the Wess–Zumino term we use the formalism introduced in [28], which emphasizes the role of morphisms between bundle gerbes, in particular between trivial bundle gerbes. According to [28], equivalence classes of morphisms $\mathcal{A} : \mathcal{I}_{\rho_1} \to \mathcal{I}_{\rho_2}$ are in natural bijection with equivalence classes of hermitian vector bundles E with connection whose curvature satisfies

$$\frac{1}{n}\mathrm{tr}(\mathrm{curv}(E)) = \rho_2 - \rho_1, \tag{B.11}$$

with n the rank of E. We write $\mathrm{Bun}(\mathcal{A})$ for the vector bundle corresponding to the morphism $\mathcal{A}$. This assignment has three important properties [28]:

(1) if the morphism $\mathcal{A}$ is invertible, then the vector bundle $\mathrm{Bun}(\mathcal{A})$ is of rank one, i.e. a line bundle; furthermore

$$\mathrm{Bun}(\mathcal{A}^{-1}) = \mathrm{Bun}(\mathcal{A})^*; \tag{B.12}$$

(2) it is compatible with the composition of morphisms,

$$\mathrm{Bun}(\mathcal{A}' \circ \mathcal{A}) = \mathrm{Bun}(\mathcal{A}) \otimes \mathrm{Bun}(\mathcal{A}') \quad \text{and} \quad \mathrm{Bun}(\mathrm{id}_{\mathcal{I}_\rho}) = 1; \tag{B.13}$$

(3) it is compatible with tensor products,

$$\mathrm{Bun}(\mathcal{A}' \otimes \mathcal{A}) = \mathrm{Bun}(\mathcal{A}) \otimes \mathrm{Bun}(\mathcal{A}'). \tag{B.14}$$

As an illustration, consider a manifold M with two bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$, and a $\mathcal{G}_1$–$\mathcal{G}_2$-bimodule $\mathcal{D} : \mathcal{G}_1 \to \mathcal{G}_2 \otimes \mathcal{I}_\omega$. Suppose we have trivializations of each of the bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$, i.e. bundle gerbe isomorphisms $\mathcal{T}_i : \mathcal{G}_i \to \mathcal{I}_{\rho_i}$. By composition, we obtain a bundle gerbe morphism

$$\widetilde{\mathcal{D}} := (\mathcal{T}_2 \otimes \mathrm{id}_{\mathcal{I}_\omega}) \circ \mathcal{D} \circ \mathcal{T}_1^{-1} : \mathcal{I}_{\rho_1} \to \mathcal{I}_{\rho_2+\omega}. \tag{B.15}$$

It corresponds to a vector bundle $E := \mathrm{Bun}(\widetilde{\mathcal{D}})$ over M. Summarizing, a gerbe bimodule together with trivializations gives a hermitian vector bundle on M with connection. Let us discuss how the vector bundle E depends on the choice of the trivializations. If $\mathcal{T}_1'$ and $\mathcal{T}_2'$ are two different choices of trivializations and $\widetilde{D}'$ is the corresponding morphism (B.15), we obtain the line bundles

$$T_i := \mathrm{Bun}(\mathcal{T}_i' \circ \mathcal{T}_i^{-1}) \tag{B.16}$$

over M, of curvature $\mathrm{curv}(T_i) = \rho_i' - \rho_i$. Then we have

$$\begin{aligned} \widetilde{\mathcal{D}} &= (\mathcal{T}_2 \otimes \mathrm{id}_{\mathcal{I}_\varpi}) \circ \mathcal{D} \circ \mathcal{T}_1^{-1} \\ &\cong (\mathcal{T}_2 \circ (\mathcal{T}_2')^{-1} \otimes \mathrm{id}_{\mathcal{I}_\varpi}) \circ (\mathcal{T}_2' \otimes \mathrm{id}_{\mathcal{I}_\varpi}) \circ \mathcal{D} \circ (\mathcal{T}_1')^{-1} \circ \mathcal{T}_1' \circ \mathcal{T}_1^{-1} \\ &= (\mathcal{T}_2 \circ (\mathcal{T}_2')^{-1} \otimes \mathrm{id}_{\mathcal{I}_\varpi}) \circ \widetilde{\mathcal{D}}' \circ \mathcal{T}_1' \circ \mathcal{T}_1^{-1}. \end{aligned} \tag{B.17}$$

Using the identification Bun of bundle gerbe morphisms with vector bundles and its properties (B.13) and (B.14) we obtain

$$E \cong T_2^* \otimes E' \otimes T_1. \tag{B.18}$$

We can apply this result in the following way to the bi-brane $(Q, \mathcal{D})$. The pull-back of the bimodule $\mathcal{D}$ along the map $\phi_S : S \to Q$ gives a $(\phi_1^*\mathcal{G}_1)|_S$-$(\phi_2^*\mathcal{G}_2)|_S$-bimodule

$$\phi_S^*\mathcal{D} : (\phi_1^*\mathcal{G}_1)\big|_S \to (\phi_2^*\mathcal{G}_2)\big|_S \otimes \mathcal{I}_{\phi_S^*\varpi}. \tag{B.19}$$

The pull-back bundle gerbes $\phi_i^*\mathcal{G}_i$ over Σ_i are trivializable for dimensional reasons. A choice $\mathcal{T}_i : \phi_i^*\mathcal{G}_i \to \mathcal{I}_\rho$ of trivializations for 2-forms ρ_i on Σ_i produces a vector bundle over S. With this vector bundle E we define

$$\mathrm{hol}_{\mathcal{G}_1,\mathcal{G}_2,\mathcal{D}}(\Sigma, S) := \exp\left(\mathrm{i}\int_{\Sigma_1}\rho_1\right)\exp\left(\mathrm{i}\int_{\Sigma_2}\rho_2\right)\mathrm{tr}(\mathrm{hol}_E(S)) \in \mathbb{C} \tag{B.20}$$

to be the holonomy in the presence of the bi-brane $(Q, \mathcal{E})$. This holonomy is the appropriate generalization of the Wess–Zumino (45) term in situations where the simplifying assumptions on the topology of the background and the bi-brane do not hold any longer.

This definition does not depend on the choice of the trivializations $\mathcal{T}_1$ and $\mathcal{T}_2$, as we shall now establish. For different choices $\mathcal{T}_1'$ and $\mathcal{T}_2'$ we obtain the line bundles T_i introduced in (B.16). Since by construction we have $\partial\Sigma_1 = S$ and $\partial\Sigma_2 = \overline{S}$, and since the curvature of the bundles T_i is $\mathrm{curv}(T_i) = \rho_i' - \rho_i$, the holonomies of T_1 and T_2 around S are given by

$$\mathrm{hol}_{T_1}(S) = \exp\left(\mathrm{i}\int_{\Sigma_1}\rho_1' - \rho_1\right) \quad \text{and} \quad (\mathrm{hol}_{T_2}(S))^{-1} = \exp\left(\mathrm{i}\int_{\Sigma_2}\rho_2' - \rho_2\right), \tag{B.21}$$

respectively. From (B.18) we obtain

$$\begin{aligned}\mathrm{tr}(\mathrm{hol}_E(S)) &= \mathrm{tr}(\mathrm{hol}_{T_2^*\otimes E'\otimes T_1}(S)) \\ &= (\mathrm{hol}_{T_2}(S))^{-1}\mathrm{tr}(\mathrm{hol}_{E'}(S))\mathrm{hol}_{T_1}(S).\end{aligned} \tag{B.22}$$

Together with (B.21) this shows the independence of number (B.20) of the choice of the trivializations.

To discuss the relation between the holonomy (B.20) and the form of the Wess–Zumino term given in Section 4, suppose there exist three-dimensional oriented submanifolds B_1 and B_2 in M_1 and M_2, respectively, and a two-dimensional oriented submanifold D of Q such that

$$\partial D = \phi_S(S), \qquad \partial B_1 = \phi_1(\Sigma_1) \cup p_1(\overline{D}) \quad \text{and} \quad \partial B_2 = \phi_2(\Sigma_2) \cup p_2(D). \tag{B.23}$$

For dimensional reasons we can choose trivializations $\mathcal{T}_i : \mathcal{G}_i|_{\partial B_i} \to \mathcal{I}_{\rho_i}$ of the two bundle gerbes over ∂B_i, thus producing a vector bundle E over D of curvature

$$\frac{1}{n}\mathrm{tr}(\mathrm{curv}(E)) = \varpi|_D + p_2^*\rho_2|_D - p_1^*\rho_1|_D. \tag{B.24}$$

The pull-backs $\phi_i^*\mathcal{T}_i : \phi_i^*\mathcal{G}_i \to \mathcal{I}_{\phi^*\rho_i}$ are trivializations as used in the definition of the holonomy (B.20), which hence becomes

$$\mathrm{hol}_{\mathcal{G}_1,\mathcal{G}_2,\mathcal{D}}(\Sigma, S) = \exp\left(\mathrm{i}\int_{\phi_1(\Sigma_1)}\rho_1\right)\exp\left(\mathrm{i}\int_{\phi_2(\Sigma_2)}\rho_2\right)\mathrm{tr}(\mathrm{hol}_E(\phi_S(S))). \tag{B.25}$$

Here the holonomy of the vector bundle E around the boundary $\phi_S(S)$ of D becomes by (B.24)

$$\mathrm{tr}(\mathrm{hol}_E(\phi_S(S))) = \mathrm{tr}(\mathrm{hol}_E(\partial D)) = \exp\left(\mathrm{i}\int_D \varpi + p_2^*\rho_2 - p_1^*\rho_1\right). \tag{B.26}$$

The holonomy of the bundle gerbe $\mathcal{G}_i|_{\partial B_i}$ around the closed surface ∂B_i is, by definition,

$$\mathrm{hol}_{\mathcal{G}_i}(\partial B_i) = \exp\left(\mathrm{i}\int_{\partial B_i}\rho_i\right) = \exp\left(\mathrm{i}\int_{\phi_i(\Sigma_i)}\rho_i \pm \mathrm{i}\int_D p_i^*\rho_i\right) \tag{B.27}$$

with a minus sign for $i = 1$ and a plus sign for $i = 2$, according to the relative orientations of D and ∂B_i in (B.23). On the other hand, we have

$$\mathrm{hol}_{\mathcal{G}_i}(\partial B_i) = \exp\left(\mathrm{i}\int_{B_i} H_i\right) \tag{B.28}$$

with H_i the curvature of $\mathcal{G}_i$. Taking the last four equalities together, we obtain

$$\exp\left(\mathrm{i}\int_{B_1} H_1 + \mathrm{i}\int_{B_2} H_2 + \mathrm{i}\int_D \varpi\right) = \mathrm{hol}_{\mathcal{G}_1,\mathcal{G}_2,\mathcal{D}}(\Sigma, S). \tag{B.29}$$

We conclude that the holonomy of the bi-brane does indeed specialize to the exponential of the Wess–Zumino term in the form given in Section 4.

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